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Paper

Global asymptotic stability of nonlinear circuits related to maximum flow problems

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Abstract: Global asymptotic stability of the nonlinear circuit similar to the one proposed by Sato *et al.* for solving the maximum flow problem is studied in this paper. The circuit consists of two independent DC voltage sources, capacitors and nonlinear resistors. It is proved rigorously that the circuit has a unique equilibrium point which is globally asymptotically stable. From the viewpoint of dynamical systems, the circuit is a cooperative system, and thus some fundamental results concerning the convergence property of cooperative systems play important roles.

Key Words: nonlinear circuit, maximum flow problem, global asymptotic stability, cooperative system, Brouwer degree

1. Introduction

There has been a long history of attempts to solve optimization problems by means of nonlinear circuits [1–12]. An advantage of this approach is that an exact or approximate optimal solution can be obtained in real time. Dennis [1] first proposed in 1959 to use circuits composed of linear resistors, DC voltage and current sources, ideal diodes and transformers to solve linear and quadratic programming problems. This idea was further extended by Stern [2]. Chua and Lin developed a canonical nonlinear programming circuit for simulating general nonlinear programming problems [3–5]. Hopfield and Tank showed that the Hopfield neural network, which can be implemented as a nonlinear circuit, is a powerful tool for solving several optimization problems such as traveling salesman problems, linear programming problems, analog-to-digital conversion, and so on [6, 7]. Kennedy and Chua [8] studied the dynamics of the modified canonical nonlinear programming circuit and showed how to guarantee the stability of the circuit. Their circuit was recently generalized by other authors so that it can be applied to a wider class of nonsmooth nonlinear programming problems [9, 10]. In addition to the above-mentioned work, various recurrent neural network models for optimization problems have been investigated in the literature (see e.g., [11, 12] and references therein). Also, some authors proposed to use SPICE, the most widely used circuit simulator, for solving constrained optimization problems [13–15]

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Recently, Sato *et al.* [16] proposed a class of nonlinear circuits[1] for solving maximum flow problems. A maximum flow problem is the problem to find the maximum amount of flow from the source to the sink through capacitated edges in a given network. Many algorithms such as Ford and Fulkerson algorithm [17] and the preflow-push algorithm [18] have been proposed for this problem. The circuit proposed by Sato *et al.* consists of two independent DC voltage sources, capacitors and nonlinear resistors. An advantage of this approach over the above-mentioned algorithms is its fast computation speed. Since the circuit converges to a steady state in a moment, it can obtain an optimal solution in real time even for very large scale problems if the steady state always corresponds to the optimal solution. On the other hand, however, this approach does not provide flexibility. The circuit for a given problem cannot be used for solving different problems. This is a major disadvantage. The dynamical behavior of the circuit is described by a set of nonlinear differential equations. It was observed through a number of computer simulations that the circuit always converges to an equilibrium point that corresponds to the maximum flow [16]. We thus expect that this property can be proved rigorously. However, unfortunately, the stability or the convergence of the circuit has not been completely understood so far. It was claimed in [16] that the circuit always converges to an equilibrium point, but the proof there is far from complete.

Since the maximum flow problem is formulated as a linear programming problem [19], all nonlinear circuits that can solve linear programming problems can be applied to the maximum flow problem. However, those circuits have in general a more complicated structure than the one proposed by Sato *et al.* [16] that is specialized to the maximum flow problem. On the other hand, a nonlinear circuit model proposed by Dennis [1] has a simple structure. However, no dynamics are involved in the model because it is composed of one DC voltage source, multiple DC current sources and ideal diodes only.

In this paper, we consider a slight modification of the nonlinear circuit proposed by Sato *et al.* [16], and prove under certain mild conditions that the circuit has a unique equilibrium point which is globally asymptotically stable. There are two main reasons why we will study the modified circuit instead of the original one. One is that the stability analysis for the former is considered to be easier than the latter due to the symmetry property of nonlinear resistors. The other is that the former presents a similar behavior to the latter for many maximum flow problems. It is thus expected that the results of this study will help us to understand the behavior of the original nonlinear circuit.

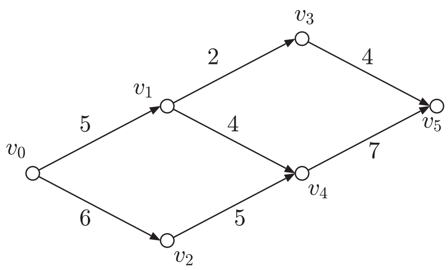
From the viewpoint of dynamical systems, the circuit considered in this paper belongs to an important class of dynamical systems called the cooperative system [20, 21]. Here, a dynamical system described by differential equations $\mathrm{d}x_i/\mathrm{d}t = F_i(x)$ $(i = 1, 2, \ldots, n)$ is called cooperative if $\partial F_i(x)/\partial x_j \geq 0$ for all $i \neq j$. Thus the proof given in this paper is based on a fundamental result [22] concerning the global stability of cooperative systems. First, the boundedness of state trajectories is proved. Second, it is proved that every equilibrium point is locally asymptotically stable. Third, the uniqueness of the equilibrium point is proved by making use of the Brouwer degree [23]. Finally, it is proved that the unique equilibrium point is globally asymptotically stable.

A preliminary version of the results of this paper was presented in NOLTA'10 [24]. However, in this conference paper, the mathematical expression of the circuit was not correct, and the proofs of some lemmas were not rigorous (see Section 4.1 for details). In the present paper, we not only provide more rigorous proofs for those lemmas but also extend the main result of [24] to a more general form. Hence the present paper is not just an extended version of the conference paper.

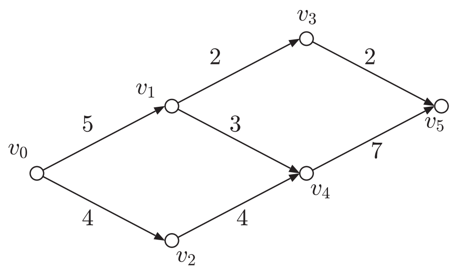
2. Maximum flow problem

First of all, we briefly review the maximum flow problem. Let $G = (V, E)$ be a simple directed graph where $V = \{v_0, v_1, \ldots, v_{n+1}\}$ is the set of vertices and $E = \{e_1, e_2, \ldots, e_m\}$ is the set of edges. An edge $e_k \in E$ directed from $v_i \in V$ to $v_j \in V$ is denoted by $e_k = (v_i, v_j)$. The set V contains two distinguished vertices: the source v_0 and the sink v_{n+1}. The source v_0 is the vertex such that E contains no edge directed to v_0. On the other hand, the sink v_{n+1} is the vertex such that E contains no edge directed from v_{n+1}. Throughout this paper, we assume:

[1] In their paper [16], the circuit is called the maximum-flow neural network.



(a)



(b)

Fig. 1. An example of the maximum flow problem [19]. (a) A simple directed graph with capacities of edges. (b) A maximum flow ($|f| = 9$).

Assumption 1 For each vertex $v_i \in V \setminus \{v_0, v_{n+1}\}$, there is at least one directed path from v_0 to v_i and there is at least one directed path from v_i to v_{n+1}.

Let $c : E \to \mathbb{R}_+$ be a capacity function where $\mathbb{R}_+$ is the set of positive numbers. The capacity of an edge (v_i, v_j) is denoted by $c(v_i, v_j)$. For any index $i \in \{1, 2, \ldots, n\}$, we define two sequences $\{N_r^-(i)\}$ and $\{N_r^+(i)\}$ of sets of indices as follows:

$$N_1^-(i) = \{j \mid (v_j, v_i) \in E\}, \quad N_r^-(i) = \{j \mid (v_j, v_k) \in E,\ k \in N_{r-1}^-(i)\} \ \ (r = 2, 3, \ldots)$$

$$N_1^+(i) = \{j \mid (v_i, v_j) \in E\}, \quad N_r^+(i) = \{j \mid (v_k, v_j) \in E,\ k \in N_{r-1}^+(i)\} \ \ (r = 2, 3, \ldots)$$

A flow on the graph G is a function $f : E \to \mathbb{R}$ satisfying the following conditions:

$$0 \le f(v_i, v_j) \le c(v_i, v_j), \quad \forall (v_i, v_j) \in E \tag{1}$$

$$\sum_{j \in N_1^-(i)} f(v_j, v_i) = \sum_{j \in N_1^+(i)} f(v_i, v_j), \quad i = 1, 2, \ldots, n \tag{2}$$

The first condition means that the flow of each edge must be less than or equal to the capacity on the edge. The second condition means that the flow must be conserved at each node, that is, the total flow into each node must be equal to the total flow out. The maximum flow problem is to find a flow f which maximizes

$$|f| \triangleq \sum_{j \in N_1^+(0)} f(v_0, v_j) = \sum_{j \in N_1^-(n+1)} f(v_j, v_{n+1}) \,.$$

A simple example of the maximum flow problem presented in [19] is shown in Fig. 1 where v_0 and v_5 are the source and the sink, respectively. The numbers beside edges in Fig. 1(a) represent their capacity and those in Fig. 1(b) represent a flow. One may easily verify that the flow in Fig. 1(b) is the maximum flow. However, the maximum flow is not uniquely determined in this example. As a matter of fact, another flow: $f(v_0, v_1) = 4$, $f(v_0, v_2) = 5$, $f(v_1, v_3) = 2$, $f(v_1, v_4) = 2$, $f(v_2, v_4) = 5$, $f(v_3, v_5) = 2$ and $f(v_4, v_5) = 7$ is also the maximum flow.

3. Nonlinear circuits for solving maximum flow problems

Sato *et al.* [16] recently proposed a method for solving maximum flow problems by making use of dynamics of nonlinear circuits. For example, the maximum flow problem in Fig. 1 can be solved by the circuit shown in Fig. 2. Now let us review how a nonlinear circuit is constructed from a given graph $G = (V, E)$. First of all, the circuit has $n+2$ nodes corresponding to $n+2$ vertices of G. Second, independent DC voltage sources X_s and X_t are connected to the nodes 0 and $n+1$, respectively. Here the value of X_s is set to be greater than that of X_t so that the current flows from the node 0 to the node $n+1$. Third, n capacitors $C_1, C_2, \ldots, C_n$ are connected to nodes $1, 2, \ldots, n$. Finally, for each pair (i, j) of nodes, a nonlinear resistor with the parameter value being equal to $c(v_i, v_j)$ is connected between these nodes if the edge (v_i, v_j) exists in G. The voltage-current characteristic of a nonlinear resistor with the parameter A is expressed as

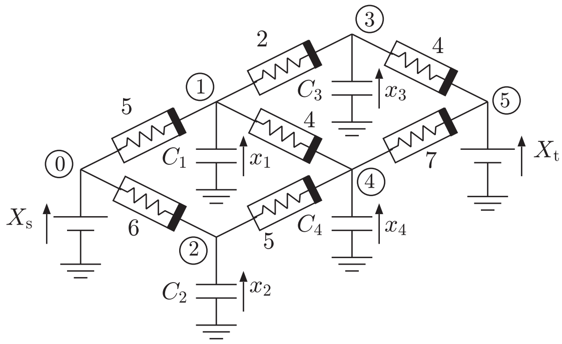


Fig. 2. A nonlinear circuit for solving the maximum flow problem in Fig. 1.

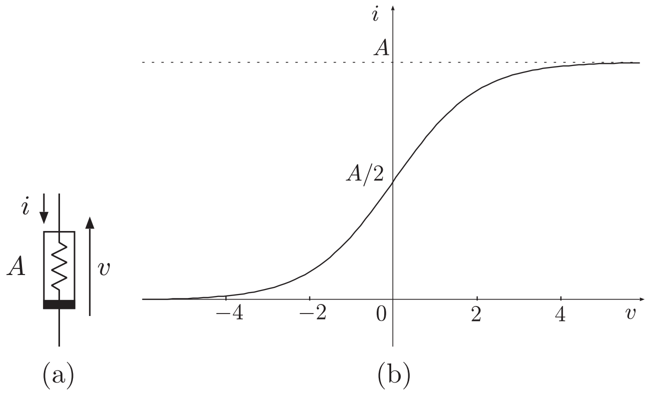


Fig. 3. Nonlinear resistor considered in [16]. (a) Symbol. (b) v-i characteristic given by $i = \sigma(v)$.

$$i = A\sigma(v) \tag{3}$$

where σ is the sigmoid function defined by

$$\sigma(y) = \frac{1}{1+\exp(-y)}$$

which is from $\mathbb{R}$ to $(0,1)$, infinitely differentiable and monotone increasing (see Fig. 3).

By taking the voltages of n capacitors as variables and denoting them $x_1, x_2, \ldots, x_n$, we obtain the set of differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{C_i}\Big[A_{0i}\sigma(X_{\rm s} - x_i) + \sum_{j=1}^{n}\{A_{ji}\sigma(x_j - x_i) - A_{ij}\sigma(x_i - x_j)\} - A_{i,n+1}\sigma(x_i - X_{\rm t})\Big]$$
$$i = 1, 2, \ldots, n \tag{4}$$

where $X_{\rm s}$ and $X_{\rm t}$ are positive constants satisfying

$$X_{\rm t} < X_{\rm s},$$

$C_1, C_2, \ldots, C_n$ are positive constants, and A_{ij}'s are nonnegative constants defined by

$$A_{ij} = \begin{cases} c(v_i, v_j), & \text{if } (v_i, v_j) \in E \\ 0, & \text{otherwise}. \end{cases}$$

Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^{\rm T}$ be any solution of the differential equation (4) and let $y_{ij}(t) = A_{ij}\sigma(x_i(t) - x_j(t))$ for all pairs of i and j such that $(v_i, v_j) \in E$. Then the collection $\{y_{ij}(t) \,|\, (v_i, v_j) \in E\}$, which represents the values of the current flowing through nonlinear resistors at time t, is a flow on the graph G for all t because the condition (1) is satisfied by the boundedness property of the sigmoid function σ and the condition (2) is satisfied by the Kirchhoff's current law. Furthermore,

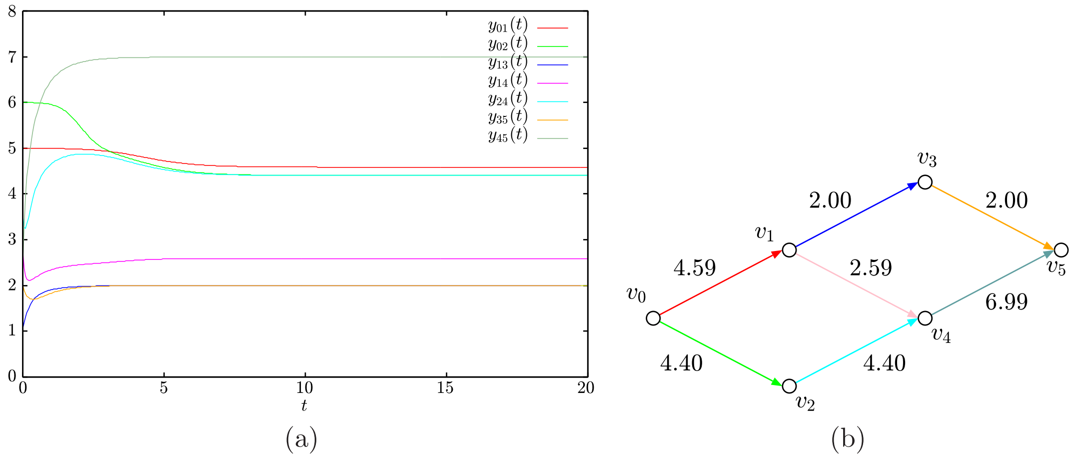


Fig. 4. Behavior of the nonlinear circuit in Fig. 2 with $X_\mathrm{s} = 10$, $X_\mathrm{t} = 0$, $C_1 = C_2 = C_3 = C_4 = 0.4$, and nonlinear resistors defined by (3). (a) Waveforms of $y_{ij}(t)$. (b) Limits of $y_{ij}(t)$.

according to the results of computer simulations carried out by Sato *et al.* [16], the solution $x(t)$ always converges and $\{\lim_{t\to\infty} y_{ij}(t) \,|\, (v_i, v_j) \in E\}$ gives an approximate solution of the maximum flow problem. Figure 4 shows an example of the waveforms $\{y_{ij}(t) \,|\, (v_i, v_j) \in E\}$ for the nonlinear circuit in Fig. 2 and their limits, which were obtained by solving (4) numerically. The obtained approximate solution satisfies $|f| = 8.99\cdots$ which is close to the maximum value.

Although nonlinear resistors defined by (3) have an important property that the current flows in one direction only, we will hereafter consider, as a simpler model, nonlinear resistors such that the voltage-current characteristic is expressed as

$$i = A\hat{\sigma}(v) \tag{5}$$

where $\hat{\sigma}$ is the sigmoid function defined by

$$\hat{\sigma}(y) = 2\sigma(y) - 1 = \frac{1 - \exp(-y)}{1 + \exp(-y)}$$

which is from $\mathbb{R}$ to $(-1, 1)$, infinitely differentiable, monotone increasing, and symmetric with respect to the origin of the v-i plane (see Fig. 5). Due to the last property, it is not necessary to distinguish two terminals of each nonlinear resistor. Furthermore, these nonlinear resistors are suitable for our theoretical analysis. To be more specific, the symmetry property that $\hat{\sigma}(-y) = -\hat{\sigma}(y)$ for all $y \in \mathbb{R}$ plays important roles in the global asymptotic stability analysis presented in the next section.

As in the case of nonlinear resistors (3), differential equations for $x_1, x_2, \ldots, x_n$, the voltages of n capacitors, are described by

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{C_i}\Big[A_{0i}\hat{\sigma}(X_\mathrm{s} - x_i) + \sum_{j=1}^{n}\{A_{ji}\hat{\sigma}(x_j - x_i) - A_{ij}\hat{\sigma}(x_i - x_j)\} - A_{i,n+1}\hat{\sigma}(x_i - X_\mathrm{t})\Big] \triangleq F_i(x), \quad i = 1, 2, \ldots, n. \tag{6}$$

Since the current can flow through each nonlinear resistor in both directions, the system (6) does not guarantee that the collection $\{y_{ij}(t) = A_{ij}\hat{\sigma}(x_i(t) - x_j(t)) \,|\, (v_i, v_j) \in E\}$ is a flow. However, the author has confirmed through computer simulations that the system (6) presents a similar behavior to (4) for many maximum flow problems. For example, Fig. 6 shows an example of the waveforms $\{y_{ij}(t) = A_{ij}\hat{\sigma}(x_i(t) - x_j(t)) \,|\, (v_i, v_j) \in E\}$ for the nonlinear circuit in Fig. 2 and their limits, which were obtained by solving (6) numerically under the same condition as in Fig. 4. The obtained approximate solution satisfies $|f| = 8.93\cdots$ which is still close to the maximum value.

4. Global asymptotic stability analysis

In this section, we will prove that the system described by the set of differential equations (6) has a unique equilibrium point which is globally asymptotically stable. Before proceeding to the proof,

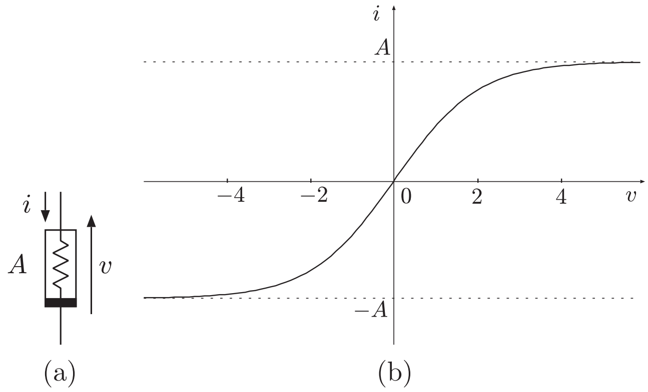


Fig. 5. Nonlinear resistor considered in this paper. (a) Symbol. (b) v-i characteristic given by $i = \hat{\sigma}(v)$.

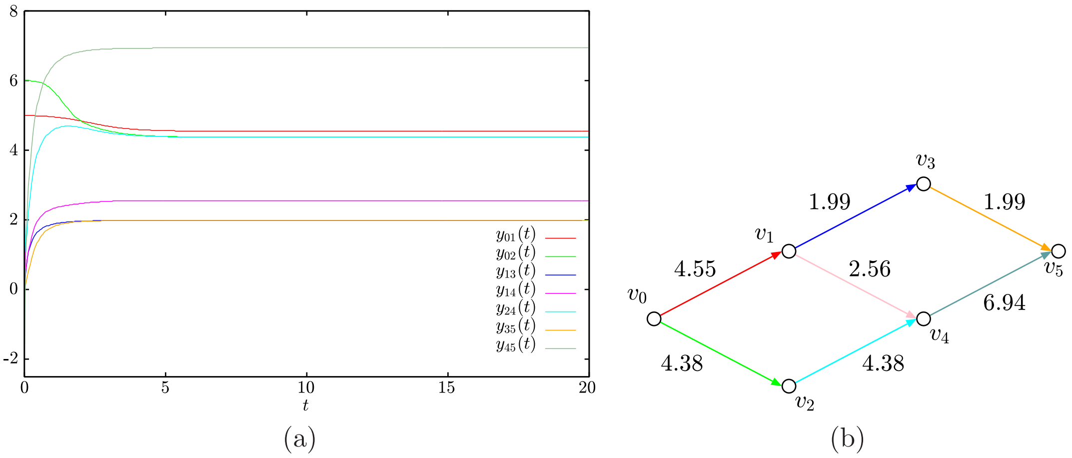


Fig. 6. Behavior of the nonlinear circuit in Fig. 2 with $X_\mathrm{s} = 10$, $X_\mathrm{t} = 0$, $C_1 = C_2 = C_3 = C_4 = 0.4$, and nonlinear resistors defined by (5). (a) Waveforms of $y_{ij}(t)$. (b) Limits of $y_{ij}(t)$.

we introduce some notation and assumptions. First, the set $[X_\mathrm{t}, X_\mathrm{s}]^n \subset \mathbb{R}^n$ is denoted by Ω, where X_s and X_t are the constants representing the voltages of the nodes 0 and $n+1$, respectively. The boundary and interior of Ω are denoted by $\partial\,\Omega$ and $\mathrm{int}\,\Omega$, respectively. Second, we assume:

Assumption 2 The initial value $x(0)$ is chosen from a bounded set $\Lambda = [L, U]^n \subset \mathbb{R}^n$ where L and U are any constants such that $L \le X_\mathrm{t}$ and $X_\mathrm{s} \le U$.

Like Ω, the boundary and interior of Λ are denoted by $\partial\,\Lambda$ and $\mathrm{int}\,\Lambda$, respectively.

4.1 Boundedness of solutions

We first prove that any solution of the system (6) is contained in the closed set $\Lambda = [L, U]^n$ defined above, i.e., Λ is a positively invariant set.

Lemma 1 Any solution $x(t)$ of the system (6) belongs to Λ for all $t \ge 0$.

Proof. It follows from Assumption 2 that $x(0) \in \Lambda$. We will thus prove that if $x(t_0) \in \Lambda$ and $x_i(t_0) = L$ ($x_i(t_0) = U$, resp.) for some $i \in \{1, 2, \ldots, n\}$ and $t_0\,(\ge 0)$ then $x_i(t)$ increases (decreases, resp.) in the time interval $(t_0, t_0 + \epsilon)$ where ϵ is a sufficiently small positive number. In the following, we focus our attention only on the case where $x_i(t_0) = L$ because the proof for the other case is similar.

The proof will be done by contradiction. Let us assume that $x(t_0) \in \Lambda$, $x_i(t_0) = L$ and $x_i(t)$ does not increase in the time interval $(t_0, t_0 + \epsilon)$. Under these assumptions, the first derivative $\mathrm{d}x_i(t_0)/\mathrm{d}t$, which is expressed as

$$\begin{aligned}\frac{\mathrm{d}x_i(t_0)}{\mathrm{d}t} &= \frac{1}{C_i}\Big[\sum_{j\in N_1^-(i)} A_{ji}\hat{\sigma}(x_j(t_0)-x_i(t_0)) - \sum_{j\in N_1^+(i)} A_{ij}\hat{\sigma}(x_i(t_0)-x_j(t_0))\Big] \\ &= \frac{1}{C_i}\Big[\sum_{j\in N_1^-(i)} A_{ji}\hat{\sigma}(x_j(t_0)-x_i(t_0)) + \sum_{j\in N_1^+(i)} A_{ij}\hat{\sigma}(x_j(t_0)-x_i(t_0))\Big]\end{aligned}$$

with $x_0(t_0) = X_\mathrm{s}$ and $x_{n+1}(t_0) = X_\mathrm{t}$, is nonnegative because $x_i(t_0) \le x_j(t_0)$ holds for all $j \in \{0,1,\ldots,n+1\}$. If $x_j(t_0) > L$ for some $j \in N_1^-(i) \cup N_1^+(i)$ then $\mathrm{d}x_i(t_0)/\mathrm{d}t$ is strictly positive and thus $x_i(t)$ increases around $t = t_0$, but this contradicts the assumption that $x_i(t)$ does not increase in the interval $(t_0, t_0+\epsilon)$. Therefore, we have

$$x_j(t_0) = L, \quad \forall j \in N_1^-(i) \cup N_1^+(i)\,.$$

Since $\mathrm{d}x_i(t_0)/\mathrm{d}t$ vanishes in this case, the second derivative $\mathrm{d}^2x_i(t_0)/\mathrm{d}t^2$, which is given by

$$\frac{\mathrm{d}^2x_i(t_0)}{\mathrm{d}t^2} = \frac{1}{C_i}\left[\sum_{j\in N_1^-(i)} A_{ji}\hat{\sigma}'(0)\frac{\mathrm{d}x_j(t_0)}{\mathrm{d}t} + \sum_{j\in N_1^+(i)} A_{ij}\hat{\sigma}'(0)\frac{\mathrm{d}x_j(t_0)}{\mathrm{d}t}\right],$$

has to be considered in order to understand the behavior of $x_i(t)$ around $t = t_0$. If $x_k(t_0) > L$ holds for some $k \in N_2^-(i) \cup N_2^+(i)$ then there exists at least one $j \in N_1^-(i) \cup N_1^+(i)$ such that $\mathrm{d}x_j(t_0)/\mathrm{d}t$ is strictly positive which implies that $\mathrm{d}^2x_i(t_0)/\mathrm{d}t^2$ is strictly positive. Thus $x_i(t)$ increases in the interval $(t_0, t_0+\epsilon)$, but this contradicts the assumption that $x_i(t)$ does not increase. Therefore, we have

$$x_k(t_0) = L, \quad \forall k \in N_2^-(i) \cup N_2^+(i)\,.$$

By repeating this argument, we have

$$x_j(t_0) = L, \quad \forall j \in N_r^-(i) \cup N_r^+(i), \quad r = 1,2,\ldots$$

However, this cannot occur because $0 \in N_r^-(i)$ holds for some r due to Assumption 1 and $x_0(t_0) = X_\mathrm{s} > L$. □

Lemma 1 is regarded as a generalization of Lemma 1 in the conference paper [24] because the initial value $x(0)$ is restricted to Ω in [24]. In other words, only a special case where $\Lambda = \Omega$ is considered. Furthermore, the proof of Lemma 1 given in [24] is incomplete because higher order derivatives were not considered.

4.2 Local asymptotic stability of an equilibrium point

We next prove that equilibrium points of the system (6) can exist only in $\mathrm{int}\,\Omega$ and that any of them is locally asymptotically stable.

Lemma 2 If $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^\mathrm{T}$ is an equilibrium point of the system (6) then $x^* \in \mathrm{int}\,\Omega$ and is locally asymptotically stable.

Proof. Suppose that the system (6) has an equilibrium point x^* such that $\min_{1\le i\le n}\{x_i^*\} \le X_\mathrm{t}$. Let i_1 be any index such that $x_{i_1}^* = \min_{1\le i\le n}\{x_i^*\}$. Then, for all $j \in N_1^-(i_1)$, x_j^* must be equal to $x_{i_1}^*$ because otherwise $F_{i_1}(x^*)$ is positive which contradicts the assumption that x^* is an equilibrium point. Similarly, for all $k \in N_2^-(i_1)$, x_k^* must be equal to $x_{i_1}^*$. By repeating this argument, we can say that there exists an integer l such that $(v_0, v_l) \in E$ and $x_l^* = x_{i_1}^* \le X_\mathrm{t}$. However, this implies that $F_l(x^*)$ is positive which leads to a contradiction. Therefore, we can conclude that if x^* is an equilibrium point then $x_i^* > X_\mathrm{t}$ for $i = 1,2,\ldots,n$. In the same way, we can also conclude that if x^* is an equilibrium point then $x_i^* < X_\mathrm{s}$ for $i = 1,2,\ldots,n$. This completes the proof of the first statement.

For the second statement, we consider the Jacobian matrix of the vector field $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^\mathrm{T}$ at an equilibrium point x^*, which is denoted by $J_F(x^*) \in \mathbb{R}^{n\times n}$. The (i,j) element of $J_F(x^*)$ is given by

$$J_{Fij}(x^*) = \begin{cases} \big[-A_{0i}\hat{\sigma}'(X_\mathrm{s} - x_i^*) - \sum_{k=1}^n \{A_{ki}\hat{\sigma}'(x_k^* - x_i^*) \\ \quad +A_{ik}\hat{\sigma}'(x_i^* - x_k^*)\} - A_{i,n+1}\hat{\sigma}'(x_i^* - X_\mathrm{t})\big]/C_i, & i = j \\ \big[A_{ji}\hat{\sigma}'(x_j^* - x_i^*) + A_{ij}\hat{\sigma}'(x_i^* - x_j^*)\big]/C_i, & i \neq j \end{cases} \tag{7}$$

Note that the nonnegative constant A_{ij} is positive if and only if $(v_i, v_j) \in E$ and that $\hat{\sigma}$ is a monotone increasing function. From these facts and Assumption 1, we see that every diagonal element of $J_F(x^*)$ is negative, every off-diagonal element of $J_F(x^*)$ is nonnegative, and $J_F(x^*)$ is irreducible (for the definition of the irreducible matrix, see [25] for example). Also, it is easily seen from (7) that $J_F(x^*)$ satisfies

$$|J_{Fii}(x^*)| \geq \sum_{j=1, j\neq i}^{n} |J_{Fij}(x^*)|, \quad i = 1, 2, \ldots, n\,.$$

In particular,

$$|J_{Fkk}(x^*)| > \sum_{j=1, j\neq k}^{n} |J_{Fkj}(x^*)|$$

holds for all k such that $(v_0, v_k) \in E$ or $(v_k, v_{n+1}) \in E$. Hence $J_F(x^*)$ is irreducibly diagonally dominant [25]. It is well known that if a square matrix is irreducibly diagonally dominant then it is nonsingular and if, in addition, its diagonal elements are negative then every eigenvalue has negative real part [25, Theorem 4.9]. Therefore, $J_F(x^*)$ is nonsingular and every eigenvalue has negative real part, which means that x^* is locally asymptotically stable. □

4.3 Uniqueness of equilibrium point

We prove here the uniqueness of equilibrium point of the system (6). The basic idea behind the proof is same as the one for [26, Lemma 2]. We first introduce a system of linear differential equations $\mathrm{d}x/\mathrm{d}t = G(x)$ such that it has a unique equilibrium point, and then apply degree theory [23] to the vector fields F and G.

Lemma 3 The system (6) has a unique equilibrium point.

Proof. Consider the system of linear differential equations:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{1}{C_i}\Big[A_{0i}(X_\mathrm{s} - x_i) + \sum_{j=1}^{n}\{A_{ji}(x_j - x_i) - A_{ij}(x_i - x_j)\} - A_{i,n+1}(x_i - X_\mathrm{t})\Big] \triangleq G_i(x),$$
$$i = 1, 2, \ldots, n\,. \tag{8}$$

Let $C = \mathrm{diag}(C_1, C_2, \ldots, C_n) \in \mathbb{R}^{n\times n}$. Let a constant matrix $K = (K_{ij}) \in \mathbb{R}^{n\times n}$ and a constant vector $b = (b_1, b_2, \ldots, b_n)^\mathrm{T} \in \mathbb{R}^n$ be defined as

$$K_{ij} = \begin{cases} -A_{0i} - \sum_{k=1}^n (A_{ki} + A_{ik}) - A_{i,n+1}, & \text{if } i = j \\ A_{ji} + A_{ij}, & \text{if } i \neq j \end{cases}$$

$$b_i = A_{0i}X_\mathrm{s} + A_{i,n+1}X_\mathrm{t}\,.$$

Then (8) can be rewritten in a matrix form as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = C^{-1}(Kx + b) \triangleq G(x) \tag{9}$$

where $G(x) = (G_1(x), G_2(x), \ldots, G_n(x))^\mathrm{T}$. By applying the same argument as in the proof of the second part of Lemma 2 to $C^{-1}K$, we can show that $C^{-1}K$ is nonsingular and every eigenvalue of $C^{-1}K$ has negative real part. In particular, K is nonsingular and every eigenvalue of K is a negative real number because K is symmetric. Hence the system (9) has a unique equilibrium point $\hat{x} = -K^{-1}b \in \mathbb{R}^n$ which is locally asymptotically stable. Moreover, we can show that $\hat{x} \in \mathrm{int}\,\Omega$ as follows. Suppose that $\hat{x} \notin \mathrm{int}\,\Omega$. Let i_1 and i_2 be any indices such that $\hat{x}_{i_1} = \min_{1\leq i\leq n}\{\hat{x}_i\}$ and $\hat{x}_{i_2} = \max_{1\leq i\leq n}\{\hat{x}_i\}$, respectively. Then, at least one of two inequalities: $\hat{x}_{i_1} \leq X_\mathrm{t}$ and $\hat{x}_{i_2} \geq X_\mathrm{s}$

holds. In the former case, by applying the same argument as in the proof of the first part of Lemma 2, we can say that there must exist an integer i such that $(v_0, v_i) \in E$ and $\hat{x}_i = \hat{x}_{i_1} \leq X_{\mathrm{t}}$. However, this leads to a contradiction. In the latter case, we can say in the same way that there must exist an integer i such that $(v_i, v_{n+1}) \in E$ and $\hat{x}_i = \hat{x}_{i_2} \geq X_{\mathrm{s}}$. However, this also leads to a contradiction. Therefore we can conclude that $\hat{x} \in \operatorname{int}\Omega$.

We next study the number of equilibrium points of the system (6) by means of degree theory [23]. The Brouwer degree [23] of the vector field $F(x)$ with respect to $\operatorname{int}\Omega$ and value 0 is defined by

$$d(F, \operatorname{int}\Omega, 0) \triangleq \sum_{x \in F^{-1}(0) \cap \operatorname{int}\Omega} \operatorname{sgn}\left(|J_F(x)|\right) .$$

By Lemma 2 we have

$$d(F, \operatorname{int}\Omega, 0) = m \times (-1)^n \tag{10}$$

where m is the number of equilibrium points of the system (6) in $\operatorname{int}\Omega$. On the other hand, the Brouwer degree of the vector field $G(x)$ with respect to $\operatorname{int}\Omega$ and value 0 is given by

$$d(G, \operatorname{int}\Omega, 0) = \operatorname{sgn}(|C^{-1}K|) = \operatorname{sgn}(|K|) = (-1)^n . \tag{11}$$

Let $H(s, x)$ be defined by

$$H(s, x) \triangleq sF(x) + (1 - s)G(x) .$$

Obviously $H(0, x) = G(x)$, $H(1, x) = F(x)$ and $H(s, x)$ is continuous on $[0, 1] \times \Omega$. Thus $H(s, x)$ is a homotopy between $F(x)$ and $G(x)$. Let us now prove that $H(s, x) \neq 0$ for all $s \in (0, 1)$ and all $x \in \partial\Omega$. If this is true, we have

$$d(F, \operatorname{int}\Omega, 0) = d(G, \operatorname{int}\Omega, 0) \tag{12}$$

from the homotopy invariance property [23], and we can conclude from Eqs. (10)–(12) that m, the number of equilibrium points of the system (6), must be one. The proof will be done by contradiction. Suppose that $H(\tilde{s}, \tilde{x}) = 0$ for some $\tilde{s} \in (0, 1)$ and $\tilde{x} \in \partial\Omega$. Then we have

$$F(\tilde{x}) = -\frac{1 - \tilde{s}}{\tilde{s}} G(\tilde{x}) . \tag{13}$$

By applying the same argument as in the first step of the proof of Lemma 1, we can easily show that there exists at least one i such that one of the following two conditions:

1) $\tilde{x}_i = X_{\mathrm{t}}$ and $G_i(\tilde{x}) > 0$

2) $\tilde{x}_i = X_{\mathrm{s}}$ and $G_i(\tilde{x}) < 0$

is satisfied. This fact together with (13) imply that there exists at least one i such that one of the following two conditions:

1) $\tilde{x}_i = X_{\mathrm{t}}$ and $F_i(\tilde{x}) < 0$

2) $\tilde{x}_i = X_{\mathrm{s}}$ and $F_i(\tilde{x}) > 0$

is satisfied. However, this contradicts Lemma 1[2]. Therefore $H(s, x) \neq 0$ for all $s \in (0, 1)$ and all $x \in \partial\Omega$. □

4.4 Global asymptotic stability of the unique equilibrium point

Now we are ready for proving our main result. A key property of the system (6), which plays an important role in the proof, is that it is a cooperative system. Cooperative systems are an important class of dynamical systems, and have been extensively studied since 1980's [20–22].

Theorem 1 The system (6) has a unique equilibrium point which belongs to $\operatorname{int}\Omega$ and is globally asymptotically stable.

[2]Consider the special case where $\Lambda = \Omega$ in Lemma 1. Then Lemma 1 says that Ω is a positively invariant set.

Proof. The system (6) is a C^1 cooperative system on Λ because

$$\frac{\partial F_i(x)}{\partial x_j} \geq 0$$

holds for all $i \neq j$ and all $x \in \Lambda$. It is known that a C^1 cooperative system in a closed box $S \subset \mathbb{R}^n$ has a globally asymptotically stable equilibrium point if and only if two conditions:

1) The system has a unique equilibrium point in S.

2) Every forward semi-orbit has compact closure in S.

hold [22, Theorem C]. Since we have already seen in Lemmas 1–3 that the system (6) satisfies these conditions with $S = \Lambda$, it has a unique equilibrium point which is globally asymptotically stable. □

5. Concluding remarks

Global asymptotic stability of a class of nonlinear circuits, which is a slight modification of nonlinear circuits proposed by Sato *et al.* for solving maximum flow problems, has been studied. It has been rigorously proved that any circuit in this class has a unique equilibrium point which is globally asymptotically stable.

However, it still remains open whether original nonlinear circuits proposed by Sato *et al.* have the same stability property. A future problem is thus to study from a theoretical point of view the validity of original nonlinear circuits for solving maximum flow problems. The first step will be to study the global asymptotic stability of the circuit. The author expects that this step can be accomplished by using the same approach as in this paper. The next step will be to make clear the relationship between the unique equilibrium point and the maximum flow of the corresponding graph. In so doing, we will first have to show that for the given voltages X_{s} and X_{t} the sum of the currents exiting the node 0, or, equivalently, the sum of the currents entering the node $n+1$, is maximized at the unique equilibrium point. We will next have to show that the sum of the currents exiting the node 0 at the unique equilibrium point will approach the maximum flow as $X_{\mathrm{s}} - X_{\mathrm{t}}$ goes to infinity.

Also, the result of this paper is restricted to nonlinear resistors such that the voltage-current characteristic is sufficiently smooth and cannot be applied to piecewise-linear resistors. Extension of the main result to the circuit with piecewise-linear resistors is another future problem.

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