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Band-Restricted Diagonally Dominant Matrices: Computational Complexity and Application

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Abstract

As a generalization of diagonally dominant matrices, we introduce a new class of square matrices called band-restricted diagonally dominant (BRDD) matrices. We prove that the problem of determining whether a given square matrix of order n is permutation-similar to some BRDD matrix is in P when the lower bandwidth l is 0 and the upper bandwidth u is $n - 1$, while the problem is NP complete when $l = 0$ and u belongs to some set of integers containing 1. We next show that a special class of BRDD matrices plays an important role in the convergence analysis of discrete-time recurrent neural networks.

Keywords: band-restricted diagonally dominant matrix, decision problem, computational complexity, neural network, convergence

1. Introduction

A real or complex square matrix is said to be diagonally dominant (DD) if the absolute value of each diagonal entry is greater than or equal to the sum of the absolute values of the offdiagonal entries in the same row. DD matrices arise in various applications such as numerical solution methods for partial differential equations [1, 2, 3] and algebraic graph theory [4, 5, 6, 7, 8]. In particular, iterative methods for solving linear systems with a nonsingular DD matrix, such as a strictly DD matrix [9, 10] and an irreducible

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DD matrix [11], have been intensively studied [2, 12, 13, 14]. Also, nearly-linear time algorithms for solving linear systems with a symmetric DD matrix approximately have recently been developed [15].

Various generalizations of DD matrices or their subclasses can be found in the literature. Feingold and Varga [16] introduced block DD matrices, which include DD matrices as a special case, and generalized the Gershgorin circle theorem by making use of some properties of these matrices. Beauwens [17] introduced semistrictly DD matrices which are a generalization of strictly DD matrices. More [18] introduced a class of nonlinear mappings as a generalization of the concept of strictly and irreducibly DD matrices. In addition to these matrices, a class of matrices called generalized strictly DD matrices has been widely studied and various criteria for determining whether a given matrix belongs to this class have been derived [19, 20, 21, 22].

In this paper, we introduce a new class of matrices called band-restricted diagonally dominant (BRDD) matrices. A real or complex square matrix is said to be BRDD if the absolute value of each diagonal entry is greater than or equal to the sum of the absolute values of the offdiagonal entries that are not only in the same row but also in a diagonally bordered band. In other words, a square matrix is BRDD for some diagonally bordered band if the matrix obtained by setting all entries outside the band to zero is DD. The class of BRDD matrices can be divided into various subclasses, each of which is characterized by the lower bandwidth denoted by l and the upper bandwidth denoted by u . One extreme case is the class of $n \times n$ BRDD matrices with $l = u = n - 1$, which corresponds to the set of all $n \times n$ DD matrices. The other extreme case is the class of $n \times n$ BRDD matrices with $l = u = 0$, which corresponds to the set of all $n \times n$ matrices.

There are two principal objectives of this paper. One is to understand the computational complexity of the problem of determining whether a given square matrix is permutation-similar to a BRDD matrix with the lower bandwidth l and the upper bandwidth u , for each pair of l and u . It is clear that the problem of determining whether a given square matrix is BRDD with the lower bandwidth l and the upper bandwidth u is in P for any pair of l and u . However, the above-mentioned problem is not so simple. In this paper, we first consider the case where $l = 0$ and $u = n - 1$, where n is the order of the given square matrix, and prove that the problem is in P. We next consider the case where $l = 0$ and $u = 1$, and prove that the problem is NP complete. These two results show that the computational complexity of the problem strongly depends on the bandwidths. We then consider the case where $l = 0$

and $\lceil(n-1)/2\rceil \leq u \leq n - \lceil n/k \rceil$ where k is any integer greater than or equal to two, and prove that the problem is NP complete.

The other objective is to show that a special class of BRDD matrices plays an important role in the convergence analysis of a recurrent neural network. Because the convergence property is very important in the applications of recurrent neural networks to information processing tasks such as associative memory [23, 24, 25] and optimization [26, 27], it has been extensively studied [28, 29, 30, 31, 32, 33, 34, 35]. In this paper, we consider a simple discrete-time recurrent neural network consisting of n neurons and prove that if the weight matrix has nonnegative diagonal entries and is permutation-similar to a BRDD matrix with $l = 0$ and $u = n - 1$ then the network converges to an equilibrium state for any initial state and bias vector. Here, it should be noted that this convergence criterion can be checked in polynomial time as stated above.

In what follows, \mathbb{C} and \mathbb{R} denote the set of complex and real numbers, respectively. \mathbb{C}^n and \mathbb{R}^n denote the set of complex and real column vectors with dimension n , respectively. $\mathbb{C}^{n \times n}$ and $\mathbb{R}^{n \times n}$ denote the set of complex and real square matrices of order n , respectively. \mathbb{Z} denotes the set of integers. $\{1, -1\}^n$ denotes the set of column vectors with dimension n whose entries are either +1 or -1.

2. Band-Restricted Diagonally Dominant Matrices

We first give a precise definition of the band-restricted diagonally dominant matrix and then provide some fundamental properties of such matrices.

Definition 1. Let n be a positive integer. Let l and u be nonnegative integers. A square matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$ is called a band-restricted diagonally dominant (BRDD) matrix with lower bandwidth l and upper bandwidth u if

$$|a_{ii}| \geq \sum_{j \in \mathcal{J}_i(n, l, u)} |a_{ij}|, \quad i = 1, 2, \dots, n \quad (1)$$

where

$$\mathcal{J}_i(n, l, u) \triangleq \{j \mid \max\{1, i-l\} \leq j \leq \min\{n, i+u\}\} \setminus \{i\}. \quad (2)$$

Notation 1. The set of all BRDD matrices of order n with lower bandwidth l and upper bandwidth u is denoted by $\mathcal{D}(n, l, u)$.

$$\mathbf{A} = \begin{pmatrix} 1 & 5 & 1 & 0 & 8 \\ 1 & 9 & 2 & 4 & 1 \\ 6 & 2 & 1 & 5 & 5 \\ 4 & 4 & 2 & 3 & 3 \\ 3 & 7 & 1 & 2 & 6 \end{pmatrix} \xrightarrow{\text{Permutation } \pi} \mathbf{B} = \begin{pmatrix} 9 & 1 & 1 & 4 & 2 \\ 7 & 6 & 3 & 2 & 1 \\ 5 & 8 & 1 & 0 & 1 \\ 4 & 3 & 4 & 3 & 2 \\ 2 & 5 & 6 & 5 & 1 \end{pmatrix}$$

Figure 1: An example of a matrix belonging to $\overline{\mathcal{D}}(5, 0, 4)$.

For example, $\mathcal{D}(n, n-1, n-1)$ is the set of all diagonally dominant square matrices of order n , and $\mathcal{D}(n, 0, 0)$ is the set of all square matrices of order n .

Definition 2. A matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$ is said to be permutation-similar to a matrix $\mathbf{B} = (b_{ij}) \in \mathbb{C}^{n \times n}$ if there exists a permutation $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that $b_{ij} = a_{\pi(i)\pi(j)}$ for all i and j .

It follows from Definitions 1 and 2 that a matrix $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$ is permutation-similar to a matrix in $\mathcal{D}(n, l, u)$ if there exists a permutation $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that

$$|a_{\pi(i)\pi(i)}| \geq \sum_{j \in \mathcal{J}_i(n, l, u)} |a_{\pi(i)\pi(j)}|, \quad i = 1, 2, \dots, n.$$

where $\mathcal{J}_i(n, l, u)$ is defined by (2).

Notation 2. The set of all matrices of order n that are permutation-similar to matrices in $\mathcal{D}(n, l, u)$ is denoted by $\overline{\mathcal{D}}(n, l, u)$.

Example 1. Let us consider the matrix \mathbf{A} shown in Fig. 1. Applying the permutation π defined by $\pi(1) = 2$, $\pi(2) = 5$, $\pi(3) = 1$, $\pi(4) = 4$ and $\pi(5) = 3$ to \mathbf{A} , we obtain the matrix \mathbf{B} shown in Fig. 1. Because \mathbf{B} belongs to $\mathcal{D}(5, 0, 4)$, that is, \mathbf{B} is a BRDD matrix with lower bandwidth 0 and upper bandwidth 4, \mathbf{A} belongs to $\overline{\mathcal{D}}(5, 0, 4)$.

Example 2. Let us consider the matrix \mathbf{A} shown in Fig. 2. Applying the permutation π defined by $\pi(1) = 3$, $\pi(2) = 6$, $\pi(3) = 1$, $\pi(4) = 4$, $\pi(5) = 5$ and $\pi(6) = 2$ to \mathbf{A} , we obtain the matrix \mathbf{B} shown in Fig. 2. Because \mathbf{B} belongs to $\mathcal{D}(6, 0, 1)$, that is, \mathbf{B} is a BRDD matrix with lower bandwidth 0 and upper bandwidth 1, \mathbf{A} belongs to $\overline{\mathcal{D}}(6, 0, 1)$.

Some fundamental properties of $\mathcal{D}(n, l, u)$ and $\overline{\mathcal{D}}(n, l, u)$ are presented in Appendix A.

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \xrightarrow{\text{Permutation } \pi} \mathbf{B} = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Figure 2: An example of a matrix belonging to $\overline{\mathcal{D}}(6, 0, 1)$.

3. A Decision Problem Related to BRDD Matrices

For a given square matrix \mathbf{A} of order n and two nonnegative integers l and u , we can check whether \mathbf{A} belongs to $\mathcal{D}(n, l, u)$ in polynomial time. However, it is not clear if the same result holds for the problem of determining whether \mathbf{A} belongs to $\overline{\mathcal{D}}(n, l, u)$. In this section, we study the computational complexity of a subclass of this problem where $l = 0$, under the assumption that arithmetic operations on complex numbers and comparisons on real numbers can be performed at unit cost.

Problem 1. Given a matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ and an integer $u \in \{1, 2, \dots, n - 1\}$, determine whether it belongs to $\overline{\mathcal{D}}(n, 0, u)$.

Let us first consider the case where $u = n - 1$. The next two lemmas give fundamental properties of $\mathcal{D}(n, 0, n - 1)$ and $\overline{\mathcal{D}}(n, 0, n - 1)$, which play an important role in the analysis of the computational complexity of Problem 1 with $u = n - 1$.

Lemma 1. *If \mathbf{A} belongs to $\mathcal{D}(n, 0, n - 1)$ then any principal submatrix of \mathbf{A} of order m belongs to $\mathcal{D}(m, 0, m - 1)$.*

PROOF. Suppose that $\mathbf{A} = (a_{ij}) \in \mathcal{D}(n, 0, n - 1)$. Let $\mathbf{A}' = (a'_{pq})$ be the matrix obtained from \mathbf{A} by deleting the $n - m$ rows and the $n - m$ columns with indices other than i_1, i_2, \dots, i_m . Assuming that $i_1 < i_2 < \dots < i_m$ without loss of generality, we have

$$|a'_{pp}| = |a_{i_p i_p}| \geq \sum_{j=i_p+1}^n |a_{i_p j}| \geq \sum_{q=p+1}^m |a_{i_p i_q}| = \sum_{q=p+1}^m |a'_{pq}|, \quad p = 1, 2, \dots, m$$

which means that $\mathbf{A}' \in \mathcal{D}(m, 0, m - 1)$. \square

Algorithm 1 Algorithm for Checking the Membership in $\overline{\mathcal{D}}(n, 0, n - 1)$

Input: $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n \times n}$

Output: TRUE or FALSE

- 1: Set $s_i \leftarrow \sum_{j=1, j \neq i}^n |a_{ij}|$ ($i = 1, 2, \dots, n$), $k \leftarrow 1$, and $\mathcal{R} \leftarrow \{1, 2, \dots, n\}$.
 - 2: Set $\mathcal{I} \leftarrow \{i \in \mathcal{R} \mid |a_{ii}| \geq s_i\}$.
 - 3: If $\mathcal{I} = \emptyset$ then return FALSE and stop. Otherwise choose an arbitrary member i_k of \mathcal{I} , set $\mathcal{R} \leftarrow \mathcal{R} \setminus \{i_k\}$ and set $s_i \leftarrow s_i - |a_{i_k i_k}|$ for all $i \in \mathcal{R}$.
 - 4: If $k = n - 1$ then return TRUE and stop. Otherwise set $k \leftarrow k + 1$ and return to Step 2.
-

Lemma 2. *If \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, n - 1)$ then any principal submatrix of \mathbf{A} of order m belongs to $\overline{\mathcal{D}}(m, 0, m - 1)$.*

PROOF. Let $\mathbf{A} = (a_{ij})$ be a matrix in $\overline{\mathcal{D}}(n, 0, n - 1)$. Then there exists a permutation π such that the matrix $\mathbf{B} = (b_{ij})$ given by $b_{ij} = a_{\pi(i)\pi(j)}$ belongs to $\mathcal{D}(n, 0, n - 1)$. Let $\mathbf{A}' \in \mathbb{C}^{m \times m}$ be the matrix obtained from \mathbf{A} by deleting the $n - m$ rows and the $n - m$ columns with indices other than i_1, i_2, \dots, i_m . Let $\{j_1, j_2, \dots, j_m\} \subseteq \{1, 2, \dots, n\}$ be the set of m distinct indices such that $\{\pi(j_1), \pi(j_2), \dots, \pi(j_m)\} = \{i_1, i_2, \dots, i_m\}$. Let $\mathbf{B}' \in \mathbb{C}^{m \times m}$ be the matrix obtained from \mathbf{B} by deleting the $n - m$ rows and the $n - m$ columns with indices other than j_1, j_2, \dots, j_m . Then, by Lemma 1, \mathbf{B}' belongs to $\mathcal{D}(m, 0, m - 1)$. Furthermore, the rows and the columns of \mathbf{B} forming \mathbf{B}' came from the rows with indices i_1, i_2, \dots, i_m and the columns with the same indices of \mathbf{A} . Therefore, \mathbf{A}' is permutation-similar to \mathbf{B}' . \square

Next we present a greedy algorithm for solving Problem 1 when $u = n - 1$, which is shown in Algorithm 1. For this algorithm, we obtain the following two results.

Lemma 3. *Algorithm 1 returns TRUE if and only if \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, n - 1)$.*

PROOF. Suppose that Algorithm 1 returns TRUE. Then the algorithm generates a sequence i_1, i_2, \dots, i_{n-1} of $n - 1$ distinct indices in $\{1, 2, \dots, n\}$. Let the remaining index be denoted by i_n . Because i_k ($k \in \{1, 2, \dots, n - 1\}$) belongs to \mathcal{I} in the k -th iteration, the following inequality holds:

$$|a_{i_k i_k}| \geq s_{i_k} = \sum_{j=1, j \neq i_k}^n |a_{i_k j}| - \sum_{l=1}^{k-1} |a_{i_k i_l}| = \sum_{l=k+1}^n |a_{i_k i_l}|.$$

Therefore, the application of the permutation $\pi(k) = i_k$ for $k = 1, 2, \dots, n$ to \mathbf{A} results in a BRDD matrix, which means that \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, n - 1)$. Suppose next that Algorithm 1 returns FALSE. Let k^* be the value of k when Algorithm 1 stops. Then, the set \mathcal{I} must be empty in the k^* -th iteration, that is, \mathbf{A} must satisfy the following conditions:

$$\forall i \in \mathcal{R}_{k^*} \triangleq \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_{k^*-1}\},$$

$$|a_{ii}| < s_i = \sum_{j=1, j \neq i}^n |a_{ij}| - \sum_{k=1}^{k^*-1} |a_{ii_k}| = \sum_{j \in \mathcal{R}_{k^*} \setminus \{i\}} |a_{ij}|.$$

This means that \mathbf{A} has a principal submatrix of order $m = n - k^* + 1$ which does not belong to $\overline{\mathcal{D}}(m, 0, m - 1)$. Therefore, by Lemma 2, we can conclude that \mathbf{A} does not belong to $\overline{\mathcal{D}}(n, 0, n - 1)$. \square

Lemma 4. *Algorithm 1 runs in $\mathcal{O}(n^2)$ time.*

PROOF. Step 1 runs in $\mathcal{O}(n^2)$ time. Steps 2, 3 and 4 run in $\mathcal{O}(n)$, $\mathcal{O}(n)$ and $\mathcal{O}(1)$ time, respectively, and the number of iterations of these three steps is at most $n - 1$. Therefore, Algorithm 1 runs in $\mathcal{O}(n^2)$ time. \square

It is worth noting that if \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, n - 1)$ then Algorithm 1 returns TRUE no matter how a member of \mathcal{I} is chosen in Step 3. The set \mathcal{I} defined in Step 2 is always non-empty because any principal submatrix of \mathbf{A} of order m belongs to $\overline{\mathcal{D}}(m, 0, m - 1)$, as shown in Lemma 2.

From Lemmas 3 and 4, we immediately obtain the following theorem.

Theorem 1. *Problem 1 is in P when $u = n - 1$.*

Let us next consider the case where $u = 1$. In this case, we obtain the following result.

Theorem 2. *Problem 1 is NP-complete when $u = 1$.*

PROOF. It is clear that the problem is in the class NP. So we prove the NP-completeness by reduction from the Hamiltonian path problem [36]. Let $G = (V, E)$ be a directed graph with the vertex set $V = \{1, 2, \dots, n\}$ and the

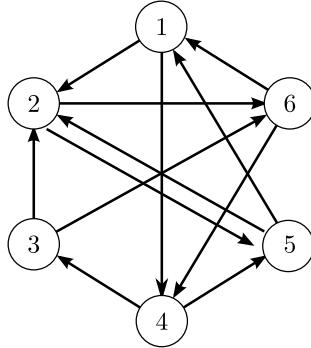


Figure 3: A directed graph having a Hamiltonian path $3 \rightarrow 6 \rightarrow 1 \rightarrow 4 \rightarrow 5 \rightarrow 2$. Applying the rule (3) to this graph, we obtain the matrix \mathbf{A} shown in Fig. 2, which belongs to $\overline{\mathcal{D}}(6, 0, 1)$.

edge set $E \subseteq V \times V$. We construct an $n \times n$ matrix $\mathbf{A} = (a_{ij}) \in \{1, 0\}^{n \times n}$ from G by the following rule:

$$a_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 0, & \text{if } i \neq j \text{ and } (i, j) \in E, \\ 1, & \text{if } i \neq j \text{ and } (i, j) \notin E. \end{cases} \quad (3)$$

It is clear that the matrix \mathbf{A} can be constructed in polynomial time. Hence we only need to show that G has a Hamiltonian path if and only if \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, 1)$. Suppose that G has a Hamiltonian path $i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_n$. An example of such a graph is shown in Fig. 3. If we define the permutation $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ as $\pi(k) = i_k$ for $k = 1, 2, \dots, n$ then we have

$$0 = |a_{\pi(k)\pi(k)}(G)| \geq |a_{\pi(k)\pi(k+1)}(G)| = 0, \quad k = 1, 2, \dots, n-1 \quad (4)$$

which means that \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, 1)$. Suppose next that G has no Hamiltonian path. Then, for any sequence of n distinct indices i_1, i_2, \dots, i_n , there exists at least one $k \in \{1, 2, \dots, n-1\}$ such that $(i_k, i_{k+1}) \notin E$. In other words, for any permutation $\pi : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$, there exists at least one $k \in \{1, 2, \dots, n-1\}$ such that $a_{\pi(k)\pi(k+1)} = 1$, which means that there is no permutation π satisfying (4). Therefore, \mathbf{A} does not belong to $\overline{\mathcal{D}}(n, 0, 1)$. \square

We finally consider the remaining case where $2 \leq u \leq n-2$.

Theorem 3. *Let k be any integer greater than or equal to two. Problem 1 is NP-complete when $\lceil(n-1)/2\rceil \leq u \leq n - \lceil n/k \rceil$.*

$$\begin{array}{c}
\text{(a)} \\
\begin{array}{ccc}
\mathbf{A} = \left(\begin{array}{cccc|cccc} & & & & X & & & \\
5 & 0 & 0 & 0 & 2 & 3 & 1 & 4 \\
10 & 5 & 0 & 0 & 2 & 3 & 1 & 4 \\
10 & 0 & 5 & 0 & 2 & 3 & 1 & 4 \\
10 & 0 & 0 & 5 & 2 & 3 & 1 & 4 \\
10 & 0 & 0 & 0 & 5 & 3 & 1 & 4 \\
10 & 0 & 0 & 0 & 2 & 5 & 1 & 4 \\
10 & 0 & 0 & 0 & 2 & 3 & 5 & 4 \\
10 & 0 & 0 & 0 & 2 & 3 & 1 & 5 \end{array} \right) & \xrightarrow{\pi} & \mathbf{B} = \left(\begin{array}{cc|cc|cc|cc} & & X_1 & & X_2 & & & \\
5 & 2 & 3 & 0 & 0 & 0 & 1 & 4 \\
10 & 5 & 3 & 0 & 0 & 0 & 1 & 4 \\
10 & 2 & 5 & 0 & 0 & 0 & 1 & 4 \\
10 & 2 & 3 & 5 & 0 & 0 & 1 & 4 \\
10 & 2 & 3 & 0 & 5 & 0 & 1 & 4 \\
10 & 2 & 3 & 0 & 0 & 5 & 1 & 4 \\
10 & 2 & 3 & 0 & 0 & 0 & 5 & 4 \\
10 & 2 & 3 & 0 & 0 & 0 & 1 & 5 \end{array} \right) \\
X_1 = \{2, 3\} & & & & X_2 = \{1, 4\} & & & \\
X_2 = \{1, 4\} & & & & & & & \end{array} \right. \\
\\
\text{(b)} \\
\begin{array}{ccc}
\mathbf{A} = \left(\begin{array}{cccc|cccc} & & & & X & & & \\
5 & 0 & 0 & 0 & 2 & 3 & 1 & 4 \\
10 & 5 & 0 & 0 & 2 & 3 & 1 & 4 \\
10 & 0 & 5 & 0 & 2 & 3 & 1 & 4 \\
10 & 0 & 0 & 5 & 2 & 3 & 1 & 4 \\
10 & 0 & 0 & 0 & 5 & 3 & 1 & 4 \\
10 & 0 & 0 & 0 & 2 & 5 & 1 & 4 \\
10 & 0 & 0 & 0 & 2 & 3 & 5 & 4 \\
10 & 0 & 0 & 0 & 2 & 3 & 1 & 5 \end{array} \right) & \xrightarrow{\pi'} & \mathbf{B}' = \left(\begin{array}{cc|cc|cc|cc} & & X_1 & & X_2 & & & \\
5 & 4 & 1 & 0 & 0 & 2 & 3 & 0 \\
10 & 5 & 1 & 0 & 0 & 2 & 3 & 0 \\
10 & 4 & 5 & 0 & 0 & 2 & 3 & 0 \\
10 & 4 & 1 & 5 & 0 & 2 & 3 & 0 \\
10 & 4 & 1 & 0 & 5 & 2 & 3 & 0 \\
10 & 4 & 1 & 0 & 0 & 5 & 3 & 0 \\
10 & 4 & 1 & 0 & 0 & 2 & 5 & 0 \\
10 & 4 & 1 & 0 & 0 & 2 & 3 & 5 \end{array} \right) \\
X_1 = \{4, 1\}, X_2 = \{2, 3\} & & & & & & & \end{array} \right.
\end{array}$$

Figure 4: Equivalence between the partition problem and Problem 1. (a) If the answer to the partition problem for X with $|X| = 4$ is “yes” then $\mathbf{A} \in \mathbb{R}^{8 \times 8}$ belongs to $\overline{\mathcal{D}}(8, 0, 4)$. (b) If \mathbf{A} belongs to $\overline{\mathcal{D}}(8, 0, 4)$ then the answer to the partition problem for X is “yes”.

PROOF. It is clear that the problem is in the class NP. So we prove the NP-completeness by reduction from the partition problem [36]. Let m be an integer greater than or equal to two. Let $X = \{x_1, x_2, \dots, x_m\}$ be a multiset of positive integers and let $s = \frac{1}{2} \sum_{j=1}^m x_i$. We choose an integer n such that

$$2m \leq n \leq km, \quad (5)$$

and construct an $n \times n$ matrix $\mathbf{A} = (a_{ij}) \in \mathbb{R}^{n \times n}$ from X as follows:

$$a_{ij} = \begin{cases} s, & \text{if } i = j, \\ 2s, & \text{if } 2 \leq i \leq n \text{ and } j = 1, \\ x_{j-n+m}, & \text{if } i \neq j \text{ and } n - m + 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

For example, the matrix $\mathbf{A} \in \mathbb{R}^{8 \times 8}$ constructed from $X = \{2, 3, 1, 4\}$ is shown in Fig. 4. It is clear that \mathbf{A} can be constructed in polynomial time. Let u be an integer satisfying

$$\lceil (n-1)/2 \rceil \leq u \leq n-m. \quad (6)$$

We hereafter show that the answer to the partition problem for the multiset X is “yes” if and only if \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, u)$. If this claim is true, the statement of the theorem holds because we have $\lceil(n-1)/2\rceil \leq u \leq n - \lceil n/k \rceil$ from (5) and (6). Suppose first that the answer to the partition problem for X is “yes”, that is, there exist two subsets X_1 and X_2 of X such that $X_1 \cap X_2 = \emptyset$, $X_1 \cup X_2 = X$ and $\sum_{x_i \in X_1} x_i = \sum_{x_j \in X_2} x_j$. We assume without loss of generality that $X_1 = \{x_1, x_2, \dots, x_p\}$ and $X_2 = \{x_{p+1}, x_{p+2}, \dots, x_m\}$ where $p \in \{1, 2, \dots, m-1\}$. In this case, applying the permutation π defined by

$$\pi(i) = \begin{cases} i + n - m - 1, & \text{if } 2 \leq i \leq p + 1, \\ i - n + m + 1, & \text{if } n - m + 1 \leq i \leq n - m + p, \\ i, & \text{otherwise,} \end{cases}$$

to \mathbf{A} , we obtain the matrix $\mathbf{B} = (b_{ij})$ with $b_{ij} = a_{\pi(i)\pi(j)}$ which is given by

$$b_{ij} = \begin{cases} s, & \text{if } i = j, \\ 2s, & \text{if } 2 \leq i \leq n \text{ and } j = 1, \\ x_{j-1}, & \text{if } i \neq j \text{ and } 2 \leq j \leq p + 1, \\ x_{j-n+m}, & \text{if } i \neq j \text{ and } n - m + p + 1 \leq j \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

An example of a pair of \mathbf{A} and \mathbf{B} is shown in Fig. 4 (a). For $i = 1, 2, \dots, p$, we have

$$\sum_{j \in \mathcal{J}_i(n, 0, u)} b_{ij} = \sum_{j=i+1}^{i+u} b_{ij} \leq \sum_{j=2}^{p+u} b_{1j} \leq \sum_{j=2}^{p+n-m} b_{1j} = \sum_{j=1}^p x_j = s = b_{ii}.$$

Also, for $i = p+1, p+2, \dots, n$, we have

$$\sum_{j \in \mathcal{J}_i(n, 0, u)} b_{ij} = \sum_{j=i+1}^{\min\{i+u, n\}} b_{ij} \leq \sum_{j=p+2}^n b_{pj} = \sum_{j=p+1}^m x_j = s = b_{ii}.$$

Therefore, the matrix \mathbf{B} belongs to $\mathcal{D}(n, 0, u)$. Suppose next that \mathbf{A} belongs to $\overline{\mathcal{D}}(n, 0, u)$. Then there exists a permutation π' such that

$$s = a_{\pi'(i)\pi'(i)} \geq \sum_{j \in \mathcal{J}_i(n, 0, u)} a_{\pi'(i)\pi'(j)}, \quad i = 1, 2, \dots, n. \quad (7)$$

An example of a pair of \mathbf{A} and $\mathbf{B}' = (b'_{ij})$ with $b'_{ij} = a_{\pi'(i)\pi'(j)}$ is shown in Fig. 4 (b). The permutation satisfies $\pi'(1) = 1$ because otherwise $\pi'(i) = 1$

for some $i \in \{2, 3, \dots, n\}$ and hence $a_{\pi'(i-1)\pi'(i)} = a_{\pi'(i-1)1} = 2s > s = a_{\pi'(i-1)\pi'(i-1)}$ holds, which contradicts (7). Also, it is easy to see from the definition of \mathbf{A} that $a_{\pi'(i)\pi'(j)} = a_{\pi'(1)\pi'(j)}$ for all i and j such that $i \neq j$ and $j \neq 1$. Taking these facts into account, we have

$$s = a_{\pi'(1)\pi'(1)} \geq \sum_{j=2}^{\min\{u+1,n\}} a_{\pi'(1)\pi'(j)} = \sum_{j=2}^{u+1} a_{\pi'(1)\pi'(j)} = \sum_{j=2}^{u+1} a_{1\pi'(j)} \quad (8)$$

and

$$\begin{aligned} s = a_{\pi'(u+1)\pi'(u+1)} &\geq \sum_{j=u+2}^{\min\{2u+2,n\}} a_{\pi'(u+1)\pi'(j)} \\ &= \sum_{j=u+2}^n a_{\pi'(u+1)\pi'(j)} = \sum_{j=u+2}^n a_{\pi'(1)\pi'(j)} = \sum_{j=u+2}^n a_{1\pi'(j)}, \end{aligned} \quad (9)$$

where $\min\{u+1, n\} = u+1$ and $\min\{2u+2, n\} = n$ follow from (6). The inequalities (8) and (9), and the equality $\sum_{j=2}^n a_{1j} = 2s$ imply that

$$s = \sum_{j=2}^{u+1} a_{1\pi'(j)} = \sum_{j=u+2}^n a_{1\pi'(j)}$$

which means that the partition problem for X is “yes”. \square

The computational complexity of Problem 1 is not completely understood by Theorems 1–3. However, these results indicate the possibility that Problem 1 may be NP-complete for all values of u except $u = n - 1$. Further studies are necessary to determine whether this conjecture is true or false.

4. Application to Discrete-Time Recurrent Neural Networks

In this section, we show that a special class of BRDD matrices plays an important role in the study of recurrent neural networks. Let us consider the discrete-time recurrent neural network (DTRNN) described by the following difference equations [25]:

$$x_i(k+1) = \begin{cases} +1, & \text{if } \sum_{j=1}^n w_{ij}x_j(k) + b_i > 0, \\ x_i(k), & \text{if } \sum_{j=1}^n w_{ij}x_j(k) + b_i = 0, \\ -1, & \text{if } \sum_{j=1}^n w_{ij}x_j(k) + b_i < 0, \end{cases} \quad i = 1, 2, \dots, n \quad (10)$$

where $x_i(k) \in \{1, -1\}$ is the state value of neuron i at time $k \in \mathbb{Z}$, $w_{ij} \in \mathbb{R}$ is the connection weight from neuron j to neuron i , and $b_i \in \mathbb{R}$ is the bias of neuron i . In the following analysis, the state values of all neurons at time k are expressed by $\mathbf{x}(k) = (x_1(k), x_2(k), \dots, x_n(k))^T \in \{1, -1\}^n$, and called the state vector at time k . Also, the connection weights among all neurons are expressed by $\mathbf{W} = (w_{ij}) \in \mathbb{R}^{n \times n}$ which is called the weight matrix, and the bias values of all neurons are expressed by $\mathbf{b} = (b_1, b_2, \dots, b_n)^T \in \mathbb{R}^n$ which is called the bias vector.

A constant vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in \{1, -1\}^n$ is called an equilibrium state of the DTRNN if

$$\mathbf{x}(k) = \boldsymbol{\alpha} \Rightarrow \mathbf{x}(k+1) = \boldsymbol{\alpha}. \quad (11)$$

The next lemma provides a necessary and sufficient condition for $\boldsymbol{\alpha} \in \{1, -1\}^n$ to be an equilibrium state of the DTRNN.

Lemma 5. *Consider a DTRNN described by (10). A vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)^T \in \{1, -1\}^n$ is an equilibrium state of the DTRNN if and only if*

$$\alpha_i \left(\sum_{j=1}^n w_{ij} \alpha_j + b_i \right) \geq 0, \quad i = 1, 2, \dots, n. \quad (12)$$

PROOF. It is easily seen from (10) and (11) that $\boldsymbol{\alpha} \in \{1, -1\}^n$ is an equilibrium point if and only if the following two conditions are satisfied:

1. $\sum_{j=1}^n w_{ij} \alpha_j + b_i \geq 0$ for all i such that $\alpha_i = +1$.
2. $\sum_{j=1}^n w_{ij} \alpha_j + b_i \leq 0$ for all i such that $\alpha_i = -1$.

These two conditions are equivalent to (12). \square

From Lemma 5, we obtain the following theorem which is the main result of this section.

Theorem 4. *Let us consider a DTRNN described by (10). If $w_{ii} \geq 0$ for $i = 1, 2, \dots, n$ and \mathbf{W} belongs to $\overline{\mathcal{D}}(n, 0, n-1)$ then the sequence $\{\mathbf{x}(k)\}_{k=0}^\infty$ converges to an equilibrium state of the DTRNN for any $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{x}(0) \in \{1, -1\}^n$.*

PROOF. We assume without loss of generality that

$$w_{ii} \geq \sum_{j=i+1}^n |w_{ij}| \quad (i = 1, 2, \dots, n-1), \quad w_{nn} \geq 0 \quad (13)$$

which corresponds to the case where $\pi(i) = i$ for $i = 1, 2, \dots, n$. We first show that neuron 1 changes its state value at most once. Suppose that there exists an integer $k^* \geq 1$ such that $x_1(k^*) = -x_1(k^* - 1)$. Then it follows from Lemma 5 that

$$x_1(k^* - 1) \left(\sum_{j=1}^n w_{1j} x_j(k^* - 1) + b_1 \right) < 0$$

from which we have

$$x_1(k^*) b_1 = -x_1(k^* - 1) b_1 > x_1(k^* - 1) \sum_{j=1}^n w_{1j} x_j(k^* - 1) \geq w_{11} - \sum_{j=2}^n |w_{1j}| \geq 0.$$

Hence we obtain

$$x_1(k^*) \left(\sum_{j=1}^n w_{1j} x_j(k^*) + b_1 \right) \geq w_{11} - \sum_{j=2}^n |w_{1j}| + x_1(k^*) b_1 > w_{11} - \sum_{j=2}^n |w_{1j}| \geq 0$$

which implies that $x_1(k^* + 1) = x_1(k^*)$. Furthermore, repeating the same argument as above, we can conclude that $x_1(k) = x_1(k^*)$ for all $k \geq k^* + 1$.

Let $k_1 \geq 0$ be an integer such that $x_1(k) = x_1(k_1)$ for all $k \geq k_1$. Then the dynamics of the remaining $n-1$ neurons for $k \geq k_1$ is equivalent to that of the $(n-1)$ -neuron DTRNN with the weight matrix

$$\mathbf{W}^{(1)} = \begin{pmatrix} w_{22} & w_{23} & \cdots & w_{2n} \\ w_{32} & w_{33} & \cdots & w_{3n} \\ \vdots & \vdots & & \vdots \\ w_{n2} & w_{n3} & \cdots & w_{nn} \end{pmatrix}$$

and the bias vector

$$\mathbf{b}^{(1)} = \begin{pmatrix} b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix} - x_1(k_1) \begin{pmatrix} w_{21} \\ w_{31} \\ \vdots \\ w_{n1} \end{pmatrix}.$$

Because $\mathbf{W}^{(1)}$ belongs to $\mathcal{D}(n-1, 0, n-2)$, neuron 2 changes its state value at most once after $k = k_1$, as we have seen above. Let $k_2 \geq k_1$ be an integer such that $x_2(k) = x_2(k_2)$ for all $k \geq k_2$. Then the dynamics of the remaining $n-2$ neurons for $k \geq k_2$ is equivalent to that of the $(n-2)$ -neuron DTRNN with the weight matrix

$$\mathbf{W}^{(2)} = \begin{pmatrix} w_{33} & w_{34} & \cdots & w_{3n} \\ w_{43} & w_{44} & \cdots & w_{4n} \\ \vdots & \vdots & & \vdots \\ w_{n3} & w_{n4} & \cdots & w_{nn} \end{pmatrix}$$

and the bias vector

$$\mathbf{b}^{(2)} = \begin{pmatrix} b_3 \\ b_4 \\ \vdots \\ b_n \end{pmatrix} - x_1(k_1) \begin{pmatrix} w_{31} \\ w_{41} \\ \vdots \\ w_{n1} \end{pmatrix} - x_2(k_2) \begin{pmatrix} w_{32} \\ w_{42} \\ \vdots \\ w_{n2} \end{pmatrix}$$

Repeating this argument, we see that there is a monotone increasing sequence $k_1 \leq k_2 \leq \cdots \leq k_{n-1}$ such that, for each $i \in \{1, 2, \dots, n-1\}$, $x_i(k) = x_i(k_i)$ for all $k \geq k_i$. Hence the dynamics of the DTRNN for $k \geq k_{n-1}$ is equivalent to that of a single neuron, which is described by

$$x_n(k+1) = \begin{cases} +1, & \text{if } w_{nn}x_n(k) + b^{(n-1)} > 0, \\ x_n(k), & \text{if } w_{nn}x_n(k) + b^{(n-1)} = 0, \\ -1, & \text{if } w_{nn}x_n(k) + b^{(n-1)} < 0, \end{cases}$$

where

$$b^{(n-1)} = b_n - \sum_{i=1}^{n-1} w_{ni}x_i(k_i).$$

So it suffices for us to show that $x_n(k)$ always converges. If $w_{nn} \geq |b^{(n-1)}|$ then we have

$$\forall \alpha_n \in \{1, -1\}, \quad \alpha_n(w_{nn}\alpha_n + b^{(n-1)}) \geq w_{nn} - |b^{(n-1)}| \geq 0$$

which means that both $\alpha_n = 1$ and $\alpha_n = -1$ are equilibrium states (see Lemma 5), and thus $x_n(k) = x_n(k_{n-1})$ for all $k \geq k_{n-1}$, whatever the value of $x_n(k_{n-1})$ is. If $|b^{(n-1)}| > w_{nn}$ (≥ 0) then we have

$$\alpha_n(w_{nn}\alpha_n + b^{(n-1)}) = \begin{cases} w_{nn} + |b^{(n-1)}| > 0, & \text{if } \alpha_n = \text{sgn}(b^{(n-1)}), \\ w_{nn} - |b^{(n-1)}| < 0, & \text{if } \alpha_n = -\text{sgn}(b^{(n-1)}), \end{cases} \quad (14)$$

where $\text{sgn}(b^{(n-1)}) = 1$ if $b^{(n-1)}$ is positive, and $\text{sgn}(b^{(n-1)}) = -1$ if $b^{(n-1)}$ is negative. Equation (14) means that $\alpha_n = \text{sgn}(b^{(n-1)})$ is an equilibrium state but $\alpha_n = -\text{sgn}(b^{(n-1)})$ is not. Therefore, if $x_n(k_{n-1}) = \text{sgn}(b^{(n-1)})$ then $x_n(k) = x_n(k_{n-1})$ for all $k \geq k_{n-1}$, otherwise $x_n(k) = -x_n(k_{n-1})$ for all $k \geq k_{n-1} + 1$. This completes the proof. \square

Example 3. Let us consider the DTRNN with

$$\mathbf{W} = \begin{pmatrix} 1 & -5 & -1 & 0 & 8 \\ -1 & 9 & -2 & -4 & -1 \\ -6 & 2 & 1 & -5 & 5 \\ 4 & -4 & -2 & 3 & -3 \\ 3 & -7 & 1 & 2 & 6 \end{pmatrix}$$

and $\mathbf{b} = (3, -4, 5, -2, 7)^T$. The weight matrix \mathbf{W} not only has nonnegative diagonal entries but also belongs to $\overline{\mathcal{D}}(n, 0, n-1)$ because if we take the absolute values of all entries then the resulting matrix is identical with the matrix \mathbf{A} shown in Fig. 1. Hence it follows from Theorem 4 that the state vector converges to an equilibrium state for any initial state. This is confirmed by the state transition diagram shown in Fig. 5. One can see that there exist five equilibrium states and every state trajectory converges to one of them.

We finally consider the number of state transitions for a DTRNN to reach an equilibrium state when $\mathbf{W} \in \overline{\mathcal{D}}(n, 0, n-1)$ and $w_{ii} \geq 0$ for $i = 1, 2, \dots, n$. In the following discussion, the notation $\boldsymbol{\alpha}^{(i)} \in \{1, -1\}^n$ introduced in the caption of Fig. 5 is used to express each state. If \mathbf{W} is diagonally dominant, the number of transitions does not exceed n . To see this, suppose that there exists a sequence of state transitions $\boldsymbol{\alpha}^{(i_0)} \rightarrow \boldsymbol{\alpha}^{(i_1)} \rightarrow \dots \rightarrow \boldsymbol{\alpha}^{(i_m)}$ such that $m > n$ and only $\boldsymbol{\alpha}^{(i_m)}$ is an equilibrium state. In each state transition, at least one neuron changes its state value because $\boldsymbol{\alpha}^{(i_0)}, \boldsymbol{\alpha}^{(i_1)}, \dots, \boldsymbol{\alpha}^{(i_{m-1})}$ are not equilibrium states. However, this is impossible because each neuron can do it only once, as shown in the proof of Theorem 4. In contrast, if $\mathbf{W} \in \overline{\mathcal{D}}(n, 0, n-1)$ is not diagonally dominant, the number of state transitions required to reach an equilibrium state can be greater than n . For example, the DTRNN with

$$\mathbf{W} = \begin{pmatrix} 10 & 2 & 2 & 2 & 2 \\ -2 & 7 & 2 & 2 & 2 \\ -2 & -2 & 7 & 2 & 2 \\ -2 & -2 & -2 & 7 & 2 \\ -2 & -2 & -2 & -2 & 7 \end{pmatrix},$$

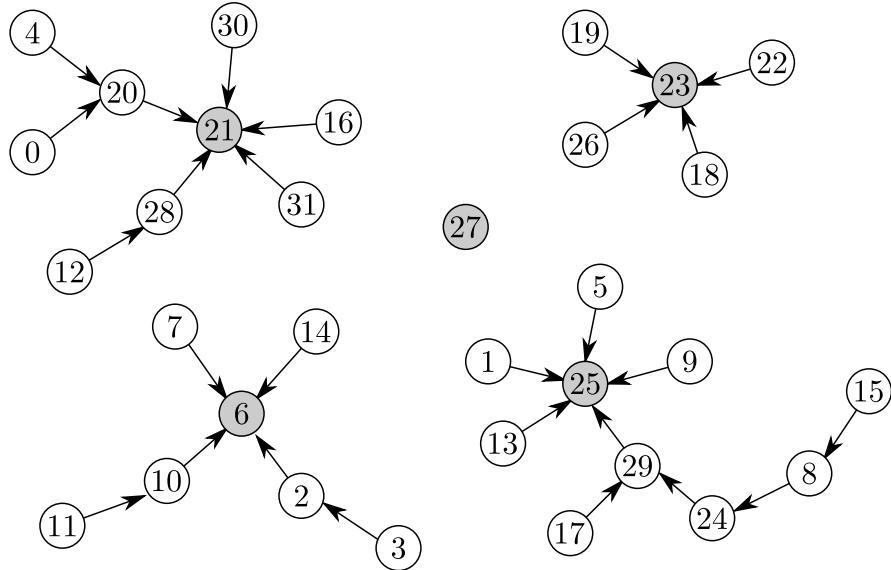


Figure 5: State transition diagram for the DTRNN in Example 3. Vertex $i \in \{0, 1, \dots, 2^n - 1\}$ represents the vector $\boldsymbol{\alpha}^{(i)} = (\alpha_1^{(i)}, \alpha_2^{(i)}, \dots, \alpha_n^{(i)})^T \in \{1, -1\}^n$ such that $i = \frac{1}{2}(\alpha_1^{(i)} + 1) \cdot 2^0 + \frac{1}{2}(\alpha_2^{(i)} + 1) \cdot 2^1 + \dots + \frac{1}{2}(\alpha_n^{(i)} + 1) \cdot 2^{n-1}$. The directed edge from vertex i to vertex j means that the state vector changes from $\boldsymbol{\alpha}^{(i)}$ to $\boldsymbol{\alpha}^{(j)}$. Vertices colored gray are equilibrium states.

which belongs to $\mathcal{D}(5, 0, 4)$ but is not diagonally dominant, and $\mathbf{b} = (3, 0, 0, 0, 0)^T$ has a sequence of nine state transitions given by

$$\begin{aligned} \boldsymbol{\alpha}^{(0)} &= (-1, -1, -1, -1, -1)^T \rightarrow \boldsymbol{\alpha}^{(16)} \rightarrow \boldsymbol{\alpha}^{(24)} \rightarrow \boldsymbol{\alpha}^{(28)} \rightarrow \boldsymbol{\alpha}^{(30)} \\ &\rightarrow \boldsymbol{\alpha}^{(31)} \rightarrow \boldsymbol{\alpha}^{(15)} \rightarrow \boldsymbol{\alpha}^{(7)} \rightarrow \boldsymbol{\alpha}^{(3)} \rightarrow \boldsymbol{\alpha}^{(1)} = (1, -1, -1, -1, -1)^T \end{aligned}$$

where $\boldsymbol{\alpha}^{(1)}$ is an equilibrium state. Also, as a generalization of this result, we can state that the DTRNN with $\mathbf{W} = (w_{ij}) \in \mathcal{D}(n, 0, n - 1)$ given by

$$w_{ij} = \begin{cases} 2n, & \text{if } i = j = 1, \\ 2n - 3, & \text{if } i = j \geq 2, \\ 2, & \text{if } i < j, \\ -2, & \text{if } i > j, \end{cases}$$

and $\mathbf{b} = (3, 0, 0, \dots, 0)^T$ has a sequence of $2n - 1$ state transitions from $\boldsymbol{\alpha}^{(0)} = (-1, -1, \dots, -1)^T$ to an equilibrium point $\boldsymbol{\alpha}^{(1)} = (1, -1, -1, \dots, -1)^T$. However, it is not clear whether this is the maximum number of state transitions for all DTRNNs with $\mathbf{W} \in \overline{\mathcal{D}}(n, 0, n - 1)$. Further studies are needed.

5. Conclusions

This paper has introduced band-restricted diagonally dominant (BRDD) matrices as a generalization of diagonally dominant matrices, and presented some theoretical results on their fundamental properties. One of the main results is about the computational complexity of the problem of determining whether a given square matrix of order n is permutation-similar to a BRDD with lower bandwidth l and upper bandwidth u . It has been proved that the problem is in P when $l = 0$ and $u = n - 1$, and NP complete when $l = 0$ and $u = 1$ and when $l = 0$ and $\lceil(n-1)/2\rceil \leq u \leq n - \lceil n/k \rceil$ where k is any integer greater than or equal to two. For other pairs of bandwidths, the computational complexity remains open for further study. Another main result is about the relationship between BRDD matrices and the convergence property of recurrent neural networks. It has been proved that the state vector of a discrete-time recurrent neural network consisting of n neurons converges to an equilibrium point for any initial state and bias vector if the weight matrix has nonnegative diagonal entries and is permutation-similar to a BRDD matrix with lower bandwidth 0 and upper bandwidth $n - 1$. Understanding the information processing capability of recurrent neural networks satisfying this condition is an important task for future research.

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Appendix A. Fundamental Properties of BRDD Matrices

Proposition 1. *If \mathbf{A} belongs to $\mathcal{D}(n, l_1, u_1)$ then \mathbf{A} belongs to $\mathcal{D}(n, l_2, u_2)$ for all pairs of l_2 and u_2 such that $0 \leq l_2 \leq l_1$ and $0 \leq u_2 \leq u_1$.*

PROOF. Let l_2 and u_2 be any integers such that $0 \leq l_2 \leq l_1$ and $0 \leq u_2 \leq u_1$. Then $\mathcal{J}_i(n, l_2, u_2) \subseteq \mathcal{J}_i(n, l_1, u_1)$ for $i = 1, 2, \dots, n$. Therefore if \mathbf{A} belongs to $\mathcal{D}(n, l_1, u_1)$ then \mathbf{A} satisfies

$$|a_{ii}| \geq \sum_{j \in \mathcal{J}_i(n, l_1, u_1)} |a_{ij}| \geq \sum_{j \in \mathcal{J}_i(n, l_2, u_2)} |a_{ij}|, \quad i = 1, 2, \dots, n$$

which means that \mathbf{A} belongs to $\mathcal{D}(n, l_2, u_2)$. \square

Proposition 2. *If \mathbf{A} belongs to $\overline{\mathcal{D}}(n, l_1, u_1)$ then \mathbf{A} belongs to $\overline{\mathcal{D}}(n, l_2, u_2)$ for all pairs of l_2 and u_2 such that $0 \leq l_2 \leq l_1$ and $0 \leq u_2 \leq u_1$.*

PROOF. Let \mathbf{A} be any matrix in $\overline{\mathcal{D}}(n, l_1, u_1)$. Then, applying a permutation π to \mathbf{A} , we obtain a matrix \mathbf{B} in $\mathcal{D}(n, l_1, u_1)$. Let l_2 and u_2 be any integers such that $0 \leq l_2 \leq l_1$ and $0 \leq u_2 \leq u_1$. Then \mathbf{B} belongs to $\mathcal{D}(n, l_2, u_2)$ due to Proposition 1. Therefore \mathbf{A} belongs to $\overline{\mathcal{D}}(n, l_2, u_2)$. \square

Proposition 3. *Let $\mathbf{A} = (a_{ij})$ belong to $\mathcal{D}(n, l, u)$. Let π be the permutation defined by $\pi(i) = n + 1 - i$ for $i = 1, 2, \dots, n$. Then the matrix $\mathbf{B} = (b_{ij})$ with $b_{ij} = a_{\pi(i)\pi(j)}$ belongs to $\mathcal{D}(n, u, l)$.*

PROOF. Because \mathbf{A} belongs to $\mathcal{D}(n, l, u)$, it satisfies (1) which is equivalent to the following inequalities:

$$|a_{\pi(i)\pi(i)}| \geq \sum_{\pi(j) \in \mathcal{J}_{\pi(i)}(n, l, u)} |a_{\pi(i)\pi(j)}|, \quad i = 1, 2, \dots, n.$$

Further, these inequalities can be rewritten as follows:

$$|b_{ii}| \geq \sum_{\pi(j) \in \mathcal{J}_{\pi(i)}(n, l, u)} |b_{ij}|, \quad i = 1, 2, \dots, n. \quad (\text{A.1})$$

Here $\pi(j) \in \mathcal{J}_{\pi(i)}(n, l, u)$ holds if and only if

$$\max\{1, n + 1 - i - l\} \leq n + 1 - j \leq \min\{n, n + 1 - i + u\}$$

and $\pi(j) \neq \pi(i)$. The first condition is equivalent to

$$\max\{1, i - u\} \leq j \leq \min\{n, i + l\}$$

and the second one is equivalent to $j \neq i$. Hence (A.1) can be rewritten as

$$|b_{ii}| \geq \sum_{j \in \mathcal{J}_i(n, u, l)} |b_{ij}|, \quad i = 1, 2, \dots, n$$

which means that \mathbf{B} belongs to $\mathcal{D}(n, u, l)$. \square

Proposition 4. *For any positive integer n and nonnegative integers l and u , $\overline{\mathcal{D}}(n, l, u) = \overline{\mathcal{D}}(n, u, l)$.*

PROOF. Let \mathbf{A} be any matrix in $\overline{\mathcal{D}}(n, l, u)$. Let π_1 be a permutation such that the matrix $\mathbf{B} = (b_{ij})$ with $b_{ij} = a_{\pi_1(i)\pi_1(j)}$ belongs to $\mathcal{D}(n, l, u)$. Let \mathbf{C} be the matrix obtained from \mathbf{B} by applying the permutation π_2 defined by $\pi_2(i) = n + 1 - i$ for $i = 1, 2, \dots, n$. Then, it follows from Proposition 3 that \mathbf{C} belongs to $\mathcal{D}(n, u, l)$. Therefore, applying the composition of π_1 and π_2 to \mathbf{A} , we obtain $\mathbf{C} \in \mathcal{D}(n, u, l)$. This means that \mathbf{A} belongs to $\overline{\mathcal{D}}(n, u, l)$. \square

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