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New Classes of Clustering Coefficient Locally Maximizing Graphs ☆

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Abstract

A simple connected undirected graph G is called a clustering coefficient locally maximizing graph if its clustering coefficient is not less than that of any simple connected graph obtained from G by rewiring an edge, that is, removing an edge and adding a new edge. In this paper, we present some new classes of clustering coefficient locally maximizing graphs. We first show that any graph composed of multiple cliques with orders greater than two sharing one vertex is a clustering coefficient locally maximizing graph. We next show that any graph obtained from a tree by replacing edges with cliques with the same order other than four is a clustering coefficient locally maximizing graph. We also extend the latter result to a more general class.

Keywords: complex network, clustering coefficient, connected caveman graph

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1. Introduction

The clustering coefficient, which was first introduced by Watts and Strogatz [1], is an important measure characterizing large and complex networks in the real world. Roughly speaking, the clustering coefficient is the probability that two vertices adjacent to a given vertex are adjacent to each other. For example, in a network of friendship between individuals, the clustering

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coefficient represents the probability that two friends of an individual will also be friends of one another [2]. It has been observed that many networks in the real world, such as the Internet, the World Wide Web, networks of coauthorship, metabolic networks and so on, exhibit a high clustering coefficient (see for example [3] and references therein). Also, it has been reported that the clustering coefficient is strongly related to the performance of Hopfield neural networks for associative memories [4], the synchronization of oscillator networks [5], the spread of behavior in online social networks [6] and the evolution of cooperation in games on networks [7, 8].

The clustering coefficient is also an important issue in the development of network models. So far, various models have been proposed in order to simulate the behavior of large and complex networks in the real world [1, 2, 9–13]. Among them, the preference attachment model proposed by Barabási and Albert [9] is one of the most well-known and widely used models because it exhibits a scale-free degree distribution, which is another important property that can be observed in many networks in the real world. However, it is known that the clustering coefficient of the Barabási and Albert model is very low [14, 15]. Therefore, based on this model, many authors have developed scale-free network models with tunable clustering coefficient [16–19]. In most of these models, the clustering coefficient can be controlled in a certain range by a user-specified parameter. On the other hand, some authors [4, 5, 20, 21] used the 2-switch [22], which rewires two edges simultaneously without changing the degree of each vertex, to increase or decrease the clustering coefficient of a network. In particular, Fukami and Takahashi [21] have recently shown experimentally that the clustering coefficient of graphs generated by the Barabási and Albert model can be increased to around 0.8 by applying the 2-switch repeatedly.

As explained above, the importance of the clustering coefficient is widely recognized in the literature. However, properties of the clustering coefficient itself have not been discussed much. To see this, let us consider the following fundamental question: What is the most clustered graph for the given number of vertices and edges? This question was first raised by Watts [23, 24]. He considered the connected caveman graph as a candidate solution and derived a general formula for its clustering coefficient. However, it is still not clear whether the connected caveman graph has the highest clustering coefficient or not.

Recently, Koizuka and Takahashi [25] studied the above-mentioned problem both theoretically and numerically. They first considered small graphs with the number of vertices being less than or equal to 10, and found a graph having the highest clustering coefficient for each possible pair of the number of vertices and the number of edges by using a brute force search. They next applied a local search algorithm to graphs with the number of vertices being less than or equal to 30, and found a graph having a high clustering coefficient for each possible pair of the number of vertices and the number of edges. Their local search algorithm is based on the edge rewiring, that is, the current graph G is replaced with a new graph G' in the neighborhood of G if G' has a higher clustering coefficient than G, where the neighborhood of G is defined as the set of all graphs that can be obtained from G by deleting an edge and adding a new edge. Although this algorithm generates a sequence $G_1, G_2, \ldots$ of graphs such that the sequence $C(G_1), C(G_2), \ldots$ of clustering coefficients is monotone increasing, it is not guaranteed that a graph with the highest clustering coefficient is always reached. In fact, we can easily find a graph G such that its clustering coefficient is higher than any graph in the neighborhood of G but is not the highest among all graphs with the same number of vertices and edges. Koizuka and Takahashi [25] thus focused their attention on such graphs that the clustering coefficient cannot be increased by the local search algorithm, which they call clustering coefficient locally maximizing graphs, and proved that any graph composed of two or three cliques sharing one vertex is a clustering coefficient locally maximizing graph.

The objective of this paper is to find more general classes of clustering coefficient locally maximizing graphs. We first show that any graph composed of multiple cliques with orders greater than two sharing one vertex is a clustering coefficient locally maximizing graph. This is a generalization of the results given by Koizuka and Takahashi [25], but our proof is much simpler than theirs. We next show that any graph obtained from a tree by replacing edges with cliques with the same order other than four is a clustering coefficient locally maximizing graph. We also extend this result to a more general class which includes graphs very similar to connected caveman graphs.

2. Notations and Definitions

Throughout this paper, by a graph, we mean a simple connected undirected graph. A graph is denoted by $G = (V(G), E(G))$ where $V(G)$ is the vertex set and $E(G)$ is the edge set. We assume that vertices of a graph

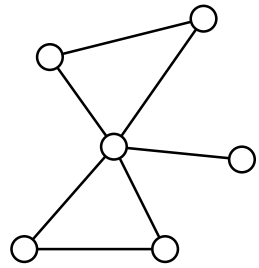

Figure 1: A clustering coefficient locally maximizing graph in $\mathcal{G}(6,7)$.

$G = (V(G), E(G))$ are always labeled by integers from 1 to $|V(G)|$. Each member of $E(G)$ is thus expressed as $\{i, j\}$ where i and j are distinct integers from 1 to $|V(G)|$. Let $\mathcal{G}(n, m)$ be the set of all graphs composed of n vertices and m edges. Apparently $\mathcal{G}(n, m)$ is non-empty if and only if $n - 1 \le m \le n(n-1)/2$.

The clustering coefficient of a graph can be defined in multiple ways [1, 3, 10]. In this paper, we focus our attention on the definition introduced by Watts and Strogatz [1]. For a given graph $G = (V(G), E(G)) \in \mathcal{G}(n, m)$, the clustering coefficient of the vertex $i \in V(G)$ is defined by

$$C_i(G) = \begin{cases} \dfrac{t_i(G)}{d_i(G)(d_i(G)-1)/2}, & \text{if } \ d_i(G) \ge 2\,, \\ 0, & \text{if } \ d_i(G) \le 1\,, \end{cases}$$

where $d_i(G)$ is the degree of the vertex i and $t_i(G)$ is the number of triangles containing the vertex i, that is,

$$t_i(G) = \left|\{\{j,k\} \in E(G) \mid \{i,j\}, \{i,k\} \in E(G)\}\right| \,.$$

The clustering coefficient of the graph $G = (V(G), E(G))$ is then defined by

$$C(G) = \frac{1}{n} \sum_{i=1}^{n} C_i(G)\,.$$

If a graph $G \in \mathcal{G}(n, m)$ satisfies $C(G) \ge C(G')$ for all $G' \in \mathcal{G}(n, m)$ then we call G a clustering coefficient maximizing graph in $\mathcal{G}(n, m)$. If a graph $G \in \mathcal{G}(n, m)$ satisfies $C(G) \ge C(G')$ for all $G' \in \mathcal{G}(n, m)$ that are obtained from G by rewiring an edge, that is, removing an edge and adding a new edge, then we call G a clustering coefficient locally maximizing graph in $\mathcal{G}(n,m)$. It is important to note that a clustering coefficient locally maximizing graph in $\mathcal{G}(n,m)$ is not necessarily a clustering coefficient maximizing graph in $\mathcal{G}(n,m)$. In fact, the graph shown in Fig. 1 is a clustering coefficient locally maximizing graph in $\mathcal{G}(6,7)$ but it is not a clustering coefficient maximizing graph in $\mathcal{G}(6,7)$ (see [25] for more details).

3. Main Results

We first show that if a graph is composed of multiple cliques with orders greater than two sharing one vertex then it is a clustering coefficient locally maximizing graph.

Theorem 1. *Let B be any integer greater than one. If the vertex set of a graph $G = (V(G), E(G)) \in \mathcal{G}(n,m)$ has a partition $\{V_0, V_1, V_2, \ldots, V_B\}$ that satisfies the following conditions then G is a clustering coefficient locally maximizing graph in $\mathcal{G}(n,m)$.*

1. $|V_b| \geq 1$ *for* $b = 1, 2, \ldots, B$.
2. *The subgraph of G induced by V_0 is the star graph S_{B+1} with the edge set $\{e_1, e_2, \ldots, e_B\}$.*
3. *The subgraph of G induced by $\bar{V}_b = V_b \cup e_b$*[1] *is complete for $b = 1, 2, \ldots, B$.*
4. *For $b = 1, 2, \ldots, B$, if $i \in V_b$ and $j \in V(G) \setminus \bar{V}_b$ then $\{i, j\} \notin E(G)$.*

Proof. Let $G \in \mathcal{G}(n,m)$ be any graph satisfying the conditions. We can assume without loss of generality that 1) $V_0 = \{1, 2, \ldots, B+1\}$, 2) the vertex $B+1$ is adjacent to all other vertices, and 3) for $b = 1, 2, \ldots, B$, the vertex b is adjacent to all vertices in V_b (See Fig. 2). In order to prove that G is a clustering coefficient locally maximizing graph, we have to consider all graphs that can be obtained from G by removing an existing edge and adding a new edge. However, without loss of generality, we can restrict ourselves to those graphs that can be obtained from G by removing an edge $\{1, \alpha\} \in E(G)$ with $\alpha \in \bar{V}_1$ and adding an edge $\{2, \beta\} \notin E(G)$ with $\beta \in V_1 \cup \{1, 3\}$. Let $G' \in \mathcal{G}(n, m-1)$ be the graph obtained from G by removing the edge $\{1, \alpha\}$,

[1]The edge e_b is considered here as a set of two vertices.

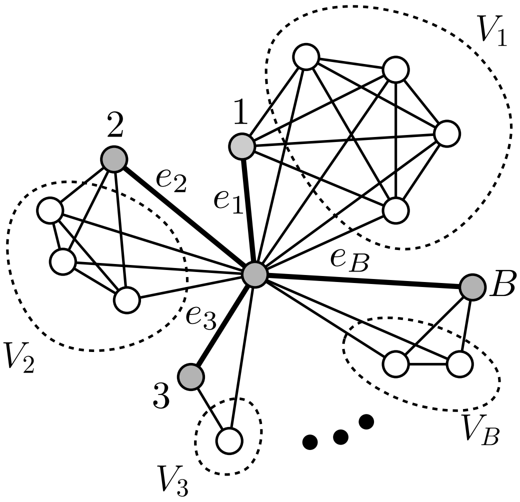

Figure 2: Structure of a graph satisfying the conditions in Theorem 1. Vertices colored gray are members of V_0.

and let $G'' \in \mathcal{G}(n, m)$ be the graph obtained from G' by adding the edge $\{2, \beta\}$. In the following, we will show that $C(G'') - C(G) = (C(G'') - C(G')) + (C(G') - C(G)) \le 0$ for any possible combination of α and β.

Let us first evaluate the quantity $C_i(G') - C_i(G)$ for each $i \in V(G)$. Depending on the value of i, there are four possible cases to be considered: (a) $i = 1$, (b) $i \in V_1$, (c) $i = B + 1$ and (d) $i \in V(G) \setminus \bar{V}_1$.

(a) $i = 1$. It is apparent that $C_1(G) = 1$. If $|V_1| \ge 2$ then $C_1(G') = 1$. If $|V_1| = 1$ then $C_1(G') = 0$ because $d_1(G') = 1$. Therefore, we have

$$C_1(G') - C_1(G) = \begin{cases} -1, & \text{if } |V_1| = 1\,, \\ 0, & \text{if } |V_1| \ge 2\,, \end{cases}$$

(b) $i \in V_1$. Suppose first that $\alpha = i$. By applying the same argument as above, we have

$$C_i(G') - C_i(G) = \begin{cases} -1, & \text{if } |V_1| = 1\,, \\ 0, & \text{if } |V_1| \ge 2\,, \end{cases}$$

Suppose next that $\alpha \ne i$. Since $t_i(G') = t_i(G) - 1$ and $d_i(G') = d_i(G)$ hold, we have

$$C_i(G') - C_i(G) \;=\; \frac{t_i(G')}{d_i(G')(d_i(G') - 1)/2} - \frac{t_i(G)}{d_i(G)(d_i(G) - 1)/2}$$

$$\begin{aligned} &= \frac{t_i(G)-1}{d_i(G)(d_i(G)-1)/2} - \frac{t_i(G)}{d_i(G)(d_i(G)-1)/2} \\ &= -\frac{1}{d_i(G)(d_i(G)-1)/2} < 0\,. \end{aligned}$$

(c) $i = B+1$. Suppose first that $\alpha = B+1$. Since $t_{B+1}(G') = t_{B+1}(G) - |V_1|$ and $d_{B+1}(G') = d_{B+1}(G) - 1$ hold, we have

$$\begin{aligned} C_{B+1}(G') - C_{B+1}(G) &= \frac{t_{B+1}(G')}{d_{B+1}(G')(d_{B+1}(G')-1)/2} \\ &\quad - \frac{t_{B+1}(G)}{d_{B+1}(G)(d_{B+1}(G)-1)/2} \\ &= \frac{t_{B+1}(G) - |V_1|}{(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \\ &\quad - \frac{t_{B+1}(G)}{d_{B+1}(G)(d_{B+1}(G)-1)/2} \\ &= \frac{2t_{B+1}(G) - d_{B+1}(G)|V_1|}{d_{B+1}(G)(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \end{aligned}$$

which can be positive. Suppose next that $\alpha \neq B+1$. Since $t_{B+1}(G') = t_{B+1}(G) - 1$ and $d_{B+1}(G') = d_{B+1}(G)$ hold, we have

$$C_{B+1}(G') - C_{B+1}(G) = -\frac{1}{d_{B+1}(G)(d_{B+1}(G)-1)/2} < 0\,.$$

(d) $i \in V(G) \setminus \bar{V}_1$. Since $t_i(G') = t_i(G)$ and $d_i(G') = d_i(G)$ hold, we have $C_i(G') - C_i(G) = 0$.

Let us next evaluate the quantity $C_i(G'') - C_i(G')$ for each $i \in V(G)$. Depending on the value of i, there are four possible cases to be considered: (a) $i = 2$, (b) $i = \beta$, (c) $i = B+1$ and (d) $i \in V(G) \setminus \{2, \beta, B+1\}$.

(a) $i = 2$. Since $C_2(G') = 1$, we have $C_2(G'') - C_2(G') = C_2(G'') - 1 \leq 0$.

(b) $i = \beta$. Let us first suppose that $d_\beta(G') \geq 2$. If $\beta \in \{1, \alpha, 3\}$ then since $C_\beta(G') = 1$ we have $C_\beta(G'') - C_\beta(G') \leq 0$. So we hereafter assume that $\beta \notin \{1, \alpha, 3\}$, that is, $\beta \in V_1$ and $\beta \neq \alpha$. Since only the vertex $B+1$ is adjacent to both 2 and β in G', we have $t_\beta(G'') = t_\beta(G') + 1$. From this equality and $d_\beta(G'') = d_\beta(G') + 1$, we have

$$C_\beta(G'') - C_\beta(G') = \frac{t_\beta(G'')}{d_\beta(G'')(d_\beta(G'')-1)/2} - \frac{t_\beta(G')}{d_\beta(G')(d_\beta(G')-1)/2}$$

$$
\begin{aligned}
&= \frac{t_\beta(G')+1}{(d_\beta(G')+1)d_\beta(G')/2} - \frac{t_\beta(G')}{d_\beta(G')(d_\beta(G')-1)/2} \\
&= \frac{-2t_\beta(G')+d_\beta(G')-1}{(d_\beta(G')+1)d_\beta(G')(d_\beta(G')-1)/2} \\
&= \frac{-d_\beta(G')(d_\beta(G')-1)+2+d_\beta(G')-1}{(d_\beta(G')+1)d_\beta(G')(d_\beta(G')-1)/2} \\
&= -\frac{1}{(d_\beta(G')+1)/2} + \frac{1}{d_\beta(G')(d_\beta(G')-1)/2}\,.
\end{aligned}
$$

Let us next suppose that $d_\beta(G')=1$. It is easily seen that this can happen only if 1) $|V_1|=1$, $\alpha=B+1$, $\beta=1$ or 2) $|V_1|=1$, $\alpha=\beta\in V_1$. In the former case, since $t_\beta(G'')=0$, we have $C_\beta(G'')=C_\beta(G')=0$. In the latter case, since $t_\beta(G'')=1$ and $d_\beta(G'')=2$, we have $C_\beta(G'')-C_\beta(G')=1$. Summarizing these discussions, we have

$$
C_\beta(G'')-C_\beta(G')
\begin{cases}
=1, & \text{if } |V_1|=1,\ \alpha=\beta\in V_1\,,\\
=-\frac{1}{(d_\beta(G')+1)/2}+\frac{1}{d_\beta(G')(d_\beta(G')-1)/2}, & \text{if } \beta\in V_1,\ \beta\neq\alpha\,,\\
\leq 0, & \text{otherwise}\,.
\end{cases}
$$

(c) $i=B+1$. It is apparent that $d_{B+1}(G'')=d_{B+1}(G')$. If $\alpha=B+1$ and $\beta=1$ then $t_{B+1}(G'')=t_{B+1}(G')+1$ holds because the vertex $B+1$ is adjacent to both 2 and β in G'. Otherwise, $t_{B+1}(G'')=t_{B+1}(G')$ holds. Therefore, we have

$$
C_{B+1}(G'')-C_{B+1}(G') = \begin{cases}
\frac{1}{d_{B+1}(G')(d_{B+1}(G')-1)/2}>0, & \text{if } \alpha=B+1 \text{ and } \beta=1\,,\\
0, & \text{otherwise}\,.
\end{cases}
$$

(d) $i\in V(G)\backslash\{2,\beta,B+1\}$. Since i is not adjacent to both 2 and β, we have $t_i(G'')=t_i(G')$. Also, it is apparent that $d_i(G'')=d_i(G')$. Therefore, we have $C_i(G'')-C_i(G')=0$.

Let us finally evaluate the quantity $C(G'')-C(G)=\frac{1}{n}\sum_{i\in V(G)}(C_i(G'')-C_i(G))$. We first consider the case where $\alpha\in V_1$ and $|V_1|=1$. In this case, we see from the analysis above that both $C_i(G')\leq C_i(G)$ and $C_i(G'')\leq C_i(G')$ hold for all $i\in V(G)\setminus\{\alpha,B+1\}$. In addition, we have $C_\alpha(G'')-C_\alpha(G)=C_\alpha(G'')-1\leq 0$ and

$$
C_{B+1}(G'')-C_{B+1}(G)
$$

$$
\begin{aligned}
&= C_{B+1}(G'') - C_{B+1}(G') + C_{B+1}(G') - C_{B+1}(G) \\
&= \frac{1}{d_{B+1}(G)(d_{B+1}(G)-1)/2} - \frac{1}{d_{B+1}(G)(d_{B+1}(G)-1)/2} \\
&= 0\,.
\end{aligned}
$$

Therefore, we can conclude that $C(G'') - C(G) \le 0$. We secondly consider the case where $\alpha \in V_1$ and $|V_1| \ge 2$. In this case, we see from the analysis above that both $C_i(G') \le C_i(G)$ and $C_i(G'') \le C_i(G')$ hold for all $i \in V(G) \setminus \{\beta, B+1\}$. As for the vertex β, $C_\beta(G'') - C_\beta(G)$ cannot be positive if $\beta \in \{1, \alpha, 3\}$. If $\beta \in V_1 \setminus \{\alpha\}$ then we have $C_\beta(G'') - C_\beta(G) = C_\beta(G'') - 1 \le 0$. As for the vertex $B+1$, $C_{B+1}(G'') - C_{B+1}(G) = 0$ holds as in the previous case. Therefore, we can conclude that $C(G'') - C(G) \le 0$. We thirdly consider the case where $\alpha = B+1$ and $|V_1| = 1$. In this case, we see from the analysis above that $C_i(G') \le C_i(G)$ holds for all $i \in V(G) \setminus \{B+1\}$. In particular, $C_i(G') - C_i(G) = -1$ holds for all $i \in \{1\} \cup V_1$. We also see that $C_i(G'') \le C_i(G')$ hold for all $i \in V(G) \setminus \{\beta, B+1\}$. We thus have

$$
\begin{aligned}
&\sum_{i \in V(G)} (C_i(G'') - C_i(G)) \\
&= \sum_{i \in V(G) \setminus \{B+1\}} (C_i(G'') - C_i(G)) + (C_{B+1}(G'') - C_{B+1}(G)) \\
&\le -2 + (C_\beta(G'') - C_\beta(G')) + (C_{B+1}(G'') - C_{B+1}(G)) \\
&\le 0
\end{aligned}
$$

which means that $C(G'') - C(G) \le 0$. We finally consider the case where $\alpha = B+1$ and $|V_1| \ge 2$. In this case, we see from the analysis above that both $C_i(G') \le C_i(G)$ and $C_i(G'') \le C_i(G')$ hold for all $i \in V(G) \setminus \{\beta, B+1\}$. As for the vertex β, we have

$$
\begin{aligned}
C_\beta(G'') - C_\beta(G) &= C_\beta(G'') - C_\beta(G') + C_\beta(G') - C_\beta(G) \\
&= -\frac{1}{(d_\beta(G')+1)/2} + \frac{1}{d_\beta(G')(d_\beta(G')-1)/2} \\
&\quad - \frac{1}{d_\beta(G)(d_\beta(G)-1)/2} \\
&= -\frac{1}{(d_\beta(G)+1)/2} \qquad (1)
\end{aligned}
$$

where we have used the equality $d_\beta(G') = d_\beta(G)$. As for the vertex $B+1$, we have

$$\begin{aligned} C_{B+1}(G'') - C_{B+1}(G) &= C_{B+1}(G'') - C_{B+1}(G') + C_{B+1}(G') - C_{B+1}(G) \\ &\leq \frac{1}{d_{B+1}(G')(d_{B+1}(G')-1)/2} \\ &\quad + \frac{2t_{B+1}(G) - d_{B+1}(G)|V_1|}{d_{B+1}(G)(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \\ &= \frac{2t_{B+1}(G) - d_{B+1}(G)(|V_1|-1)}{d_{B+1}(G)(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \qquad (2) \end{aligned}$$

where we have used the equality $d_{B+1}(G') = d_{B+1}(G) - 1$. Furthermore, substituting the equalities $d_{B+1}(G) = \sum_{b=1}^{B} d_b(G)$, $t_{B+1}(G) = \sum_{b=1}^{B} d_b(G)(d_b(G) - 1)/2$ and $|V_1| = d_1(G) - 1$ into the numerator of (2), we have

$$\begin{aligned} & 2t_{B+1}(G) - d_{B+1}(G)(|V_1|-1) \\ &= \sum_{b=1}^{B} d_b(G)(d_b(G)-1) - \sum_{b=1}^{B} d_b(G)(d_1(G)-2) \\ &= \sum_{b=1}^{B} d_b(G)(d_b(G)+1-d_1(G)) \\ &\leq \sum_{b=1}^{B} d_b(G)(d_b(G)-2) \\ &\leq \sum_{b=1}^{B} d_b(G) \sum_{b=1}^{B} (d_b(G)-2) \\ &= d_{B+1}(G)(d_{B+1}(G)-2B) \qquad (3) \end{aligned}$$

where we have used the inequality $d_1(G) = |V_1| + 1 \geq 3$. It follows from (1), (2) and (3) that

$$\begin{aligned} & \sum_{i \in V(G)} (C_i(G'') - C_i(G)) \\ &\leq C_\beta(G'') - C_\beta(G) + C_{B+1}(G'') - C_{B+1}(G) \\ &\leq -\frac{1}{(d_\beta(G)+1)/2} + \frac{d_{B+1}(G)-2B}{(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \\ &\leq -\frac{1}{(d_{B+1}(G)-1)/2} + \frac{d_{B+1}(G)-2B}{(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \end{aligned}$$

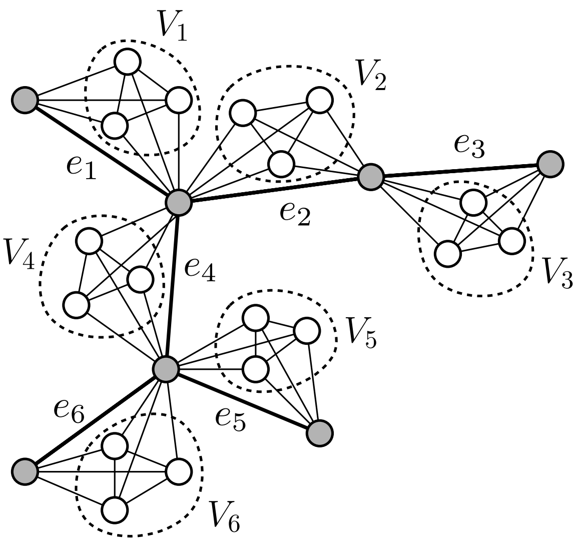

Figure 3: Structure of a graph satisfying the conditions in Theorem 2 with $B = 6$ and $K = 3$. Vertices colored gray are members of V_0.

$$
\begin{aligned}
&= -\frac{2(B-1)}{(d_{B+1}(G)-1)(d_{B+1}(G)-2)/2} \\
&< 0
\end{aligned}
$$

where the second and third inequalities follow from $d_\beta(G) \leq d_{B+1}(G)-2$ and $B \geq 2$, respectively. Therefore, we can conclude that $C(G'') - C(G) \leq 0$. □

Theorem 1 is a generalization of the results given by Koizuka and Takahashi [25, Theorems 1 and 2] in which B is restricted to either two or three. In addition, the proof of Theorem 1 is much simpler than their proofs.

We next show that if a graph is obtained from a tree by replacing its edges with cliques with the same order other than four then it is a clustering coefficient locally maximizing graph.

Theorem 2. *Let B be any integer greater than one. Let K be any positive integer other than two. If the vertex set of a graph $G = (V(G), E(G)) \in \mathcal{G}(n, m)$ has a partition $\{V_0, V_1, V_2, \ldots, V_B\}$ that satisfies the following conditions then G is a clustering coefficient locally maximizing graph in $\mathcal{G}(n, m)$.*

1. *$|V_b| = K$ for $b = 1, 2, \ldots, B$.*

2. *The subgraph of G induced by V_0 is a tree with the edge set $\{e_1, e_2, \ldots, e_B\}$.*
3. *The subgraph of G induced by $\bar{V}_b = V_b \cup e_b$ is complete for $b = 1, 2, \ldots, B$.*
4. *For $b = 1, 2, \ldots, B$, if $i \in V_b$ and $j \in V(G) \setminus \bar{V}_b$ then $\{i, j\} \notin E(G)$.*

Before proceeding to the proof of Theorem 2, we will show a fundamental property of the graphs under consideration.

Lemma 1. *Let G be any graph satisfying the conditions in Theorem 2. Then $t_i(G) = Kd_i(G)/2$ holds for any vertex $i \in V(G)$.*

Proof. If $i \in V(G) \setminus V_0$ then we have

$$t_i(G) = \frac{(K+1)K}{2} = \frac{Kd_i(G)}{2}.$$

If $i \in V_0$ then, by taking into account the fact that the degree of i in the subgraph of G induced by V_0 is given by $d_i(G)/(K+1)$, we have

$$t_i(G) = \frac{(K+1)K}{2} \cdot \frac{d_i(G)}{K+1} = \frac{Kd_i(G)}{2}$$

which completes the proof. □

Proof of Theorem 2. Let $G \in \mathcal{G}(n, m)$ be any graph satisfying the conditions. The graph shown in Fig. 3 is an example of G. Let $G' \in \mathcal{G}(n, m-1)$ be the graph obtained from G by removing an edge $\{\alpha, \beta\} \in E(G)$, and let $G'' \in \mathcal{G}(n, m)$ be the graph obtained from G' by adding an edge $\{\gamma, \delta\}$ which is neither a member of $E(G')$ nor equal to $\{\alpha, \beta\}$. In the following, we will show that $C(G'') - C(G) = (C(G'') - C(G')) + (C(G') - C(G)) \le 0$ for any possible combination of α, β, γ and δ under the assumption that $K \ge 3$. The case where $K = 1$ will be considered in Appendix.

Let us first evaluate the quantity $C_i(G') - C_i(G)$ for each $i \in V(G)$. Depending on the value of i, there are three possible cases to be considered: (a) $i \in \{\alpha, \beta\}$, (b) $i \notin \{\alpha, \beta\}$ and i is adjacent to both α and β in G, and (c) $i \notin \{\alpha, \beta\}$ and i is not adjacent to both α and β in G.

(a) $i \in \{\alpha, \beta\}$. Since $d_i(G') = d_i(G) - 1 \ge 3$ and $t_i(G') = t_i(G) - K$ hold in this case, we have

$$C_i(G') - C_i(G) = \frac{t_i(G')}{d_i(G')(d_i(G')-1)/2} - \frac{t_i(G)}{d_i(G)(d_i(G)-1)/2}$$

$$
\begin{aligned}
&= \frac{t_i(G) - K}{(d_i(G)-1)(d_i(G)-2)/2} - \frac{t_i(G)}{d_i(G)(d_i(G)-1)/2} \\
&= \frac{-Kd_i(G) + 2t_i(G)}{d_i(G)(d_i(G)-1)(d_i(G)-2)/2} \qquad (4)
\end{aligned}
$$

where the numerator vanishes because of Lemma 1. Therefore, $C_i(G') - C_i(G) = 0$.

(b) $i \notin \{\alpha, \beta\}$ and i is adjacent to both α and β in G. Since $d_i(G') = d_i(G)$ and $t_i(G') = t_i(G) - 1$, we have

$$
C_i(G') - C_i(G) = -\frac{1}{d_i(G)(d_i(G)-1)/2} < 0\,. \qquad (5)
$$

It should be noted here that there exists at least one $i \in V(G) \setminus V_0$ satisfying the above condition because $K \geq 3$. For such a vertex i, we have

$$
C_i(G') - C_i(G) = -\frac{1}{(K+1)K/2}\,.
$$

(c) $i \notin \{\alpha, \beta\}$ and i is not adjacent to both α and β in G. Since $d_i(G') = d_i(G)$ and $t_i(G') = t_i(G)$, we have $C_i(G') - C_i(G) = 0$.

Let us next evaluate the quantity $C_i(G'') - C_i(G')$ for each $i \in V(G)$. Depending on the value of i, there are three possible cases to be considered: (a) $i \in \{\gamma, \delta\}$, (b) $i \notin \{\gamma, \delta\}$ and i is adjacent to both γ and δ in G' and (c) $i \notin \{\gamma, \delta\}$ and i is not adjacent to both γ and δ in G'.

(a) $i \in \{\gamma, \delta\}$. Note that G' has at most one vertex which is adjacent to both γ and δ. This implies that $t_i(G'') \leq t_i(G') + 1$. By using this inequality and $d_i(G'') = d_i(G') + 1$, we have

$$
\begin{aligned}
C_i(G'') - C_i(G') &= \frac{t_i(G'')}{d_i(G'')(d_i(G'')-1)/2} - \frac{t_i(G')}{d_i(G')(d_i(G')-1)/2} \\
&\leq \frac{t_i(G') + 1}{(d_i(G')+1)d_i(G')/2} - \frac{t_i(G')}{d_i(G')(d_i(G')-1)/2} \\
&= \frac{-2t_i(G') + d_i(G') - 1}{(d_i(G')+1)d_i(G')(d_i(G')-1)/2}\,.
\end{aligned}
$$

Let us focus our attention on the numerator. From Lemma 1 and the analysis of the quantity $C_i(G') - C_i(G)$ for $i \in \{\alpha, \beta\}$, we see that

$$
t_i(G') \geq t_i(G) - K = \frac{Kd_i(G)}{2} - K = \frac{K(d_i(G)-2)}{2}\,.
$$

By using this inequality and the inequality $d_i(G') \le d_i(G)$, we have

$$\begin{aligned}-2t_i(G') + d_i(G') - 1 &\le -K(d_i(G) - 2) + d_i(G) - 1 \\ &= -(K-1)(d_i(G) - 2) + 1\end{aligned}$$

which is negative because $K \ge 3$ and $d_i(G) \ge 4$. Therefore, $C_i(G'') - C_i(G') < 0$ holds.

(b) $i \notin \{\gamma, \delta\}$ and i is adjacent to both γ and δ in G'. Since $t_i(G'') = t_i(G') + 1$ and $d_i(G'') = d_i(G')$ hold, we have

$$C_i(G'') - C_i(G') = \frac{1}{d_i(G')(d_i(G') - 1)/2} > 0\,. \tag{6}$$

As discussed in Case (a), G' has at most one vertex which is adjacent to both γ and δ. Strictly speaking, if there exist two distinct indices $b_1, b_2 \in \{1, 2, \ldots, B\}$ such that $\gamma \in \bar{V}_{b_1}$, $\delta \in \bar{V}_{b_2}$, and $\bar{V}_{b_1} \cap \bar{V}_{b_2} \ne \emptyset$ then such a vertex exists and is uniquely determined as the unique element of $\bar{V}_{b_1} \cap \bar{V}_{b_2} (\subset V_0)$. Otherwise, such a vertex does not exist. From this observation, we have

$$d_i(G') \ge d_i(G) - 1 \ge 2(K+1) - 1 = 2K + 1$$

where $d_i(G) \ge 2(K+1)$ follows from the fact that the vertex i of the subgraph of G induced by V_0 has a degree at least two. Therefore, (6) is bounded from above as follows:

$$C_i(G'') - C_i(G') \le \frac{1}{K(2K+1)}\,.$$

(c) $i \notin \{\gamma, \delta\}$ and i is not adjacent to both γ and δ in G'. Since $d_i(G'') = d_i(G')$ and $t_i(G'') = t_i(G')$, we have $C_i(G'') - C_i(G') = 0$.

From the analysis above, we have

$$\begin{aligned}C(G'') - C(G) &= (C(G'') - C(G')) - (C(G') - C(G)) \\ &\le \frac{1}{n}\left\{-\frac{1}{K(K+1)/2} + \frac{1}{K(2K+1)}\right\} \\ &= -\frac{1}{n} \cdot \frac{3K+1}{K(K+1)(2K+1)} \\ &< 0\end{aligned}$$

for any choice of edges $\{\alpha, \beta\}$ and $\{\gamma, \delta\}$. This means that G is a clustering coefficient locally maximizing graph in $\mathcal{G}(n, m)$. $\square$

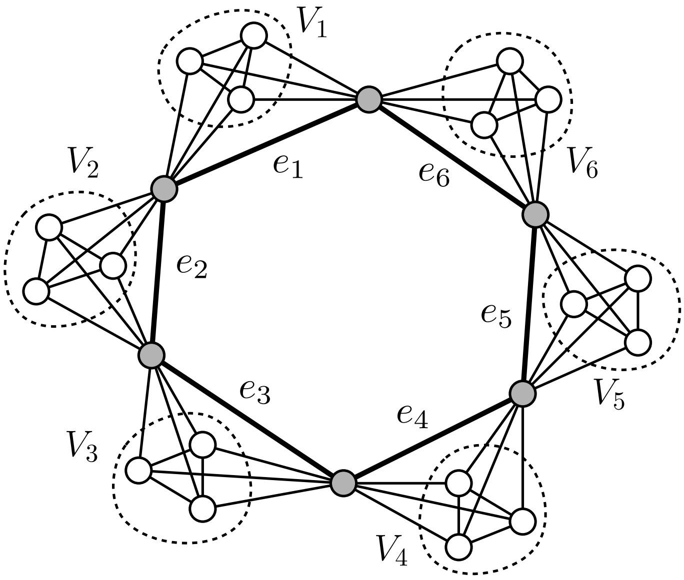

Figure 4: A graph satisfying the conditions in Theorem 3 with $B = 6$ and $K = 3$. Vertices colored gray are members of V_0. The subgraph induced by V_0 is a cycle graph with length six.

The most significant difference between Theorems 1 and 2 is that the subgraph of G induced by V_0 is restricted to a star graph in the former while it can be any tree in the latter. More specifically, the existence of a vertex which is adjacent to all other vertices is necessary for Theorem 1 but not for Theorem 2. On the other hand, the first condition that $|V_b|$ must take the same value $K(\neq 2)$ for $b = 1, 2, \ldots, B$ of Theorem 2 is more restrictive than that of Theorem 1. Therefore, Theorem 2 is not a complete generalization of Theorem 1. We will discuss the first condition of Theorem 2 in more details in Section 4.

Theorem 2 can be extended to more general setting as follows.

Theorem 3. *Let B be any integer greater than one. Let K be any positive integer other than two. If the vertex set of a graph $G = (V(G), E(G)) \in \mathcal{G}(n, m)$ has a partition $\{V_0, V_1, V_2, \ldots, V_B\}$ that satisfies the following conditions then G is a clustering coefficient locally maximizing graph in $\mathcal{G}(n, m)$.*

1. $|V_b| = K$ *for* $b = 1, 2, \ldots, B$.
2. *The subgraph of G induced by V_0 is a graph with the edge set* $\{e_1, e_2,$

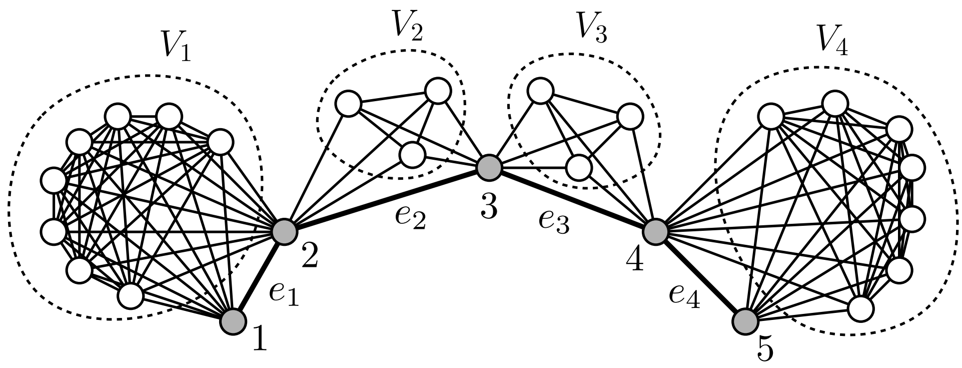

Figure 5: Graph G_1. If $|V_1|$ and $|V_4|$ are sufficiently large then G_1 is not a clustering coefficient locally maximizing graph.

$\ldots, e_B\}$ which does not contain cycles with length less than or equal to four.

3. *The subgraph of G induced by $\bar{V}_b = V_b \cup e_b$ is complete for $b = 1, 2, \ldots, B$.*
4. *For $b = 1, 2, \ldots, B$, if $i \in V_b$ and $j \in V(G) \setminus \bar{V}_b$ then $\{i, j\} \notin E(G)$.*

We omit the proof of Theorem 3 because it is similar to that of Theorem 2. One thing to note here is that if the subgraph of G induced by V_0 does not contain cycles with length less than or equal to four then it is guaranteed as in the case of Theorem 2 that G' has at most one vertex which is adjacent to both γ and δ.

An example of a graph that satisfies the condition in Theorem 3 is shown in Fig. 4. It has a very similar structure to the connected caveman graph [23].

4. Remarks on Conditions of Theorem 2

As stated in the previous section, Theorem 2 is not a complete generalization of Theorem 1 because the first condition that $|V_b| = K(\neq 2)$ for $b = 1, 2, \ldots, B$ of Theorem 2 is more restrictive than that of Theorem 1. In order to show that this condition is really necessary for Theorem 2 to hold, we consider two illustrative examples. As the first example, let us consider the graph $G_1 = (V(G_1), E(G_1))$ shown in Fig. 5 where we assume that $|V_2| = |V_3| = 3$ and $|V_1|$ and $|V_4|$ are greater than 3. Hence G_1 does not satisfy the first condition in Theorem 2. Let G_1'' be the graph obtained

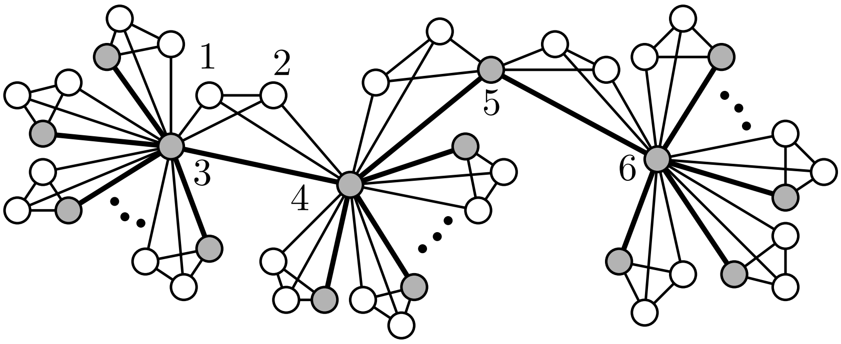

Figure 6: Graph G_2. If $d_3(G_2)$, $d_4(G_2)$ and $d_6(G_2)$ are sufficiently large then G_2 is not a clustering coefficient locally maximizing graph.

from G_1 by removing the edge $e_1 = \{1,2\}$ and adding the edge $\{2,4\}$. Then $C_i(G_1'') - C_i(G_1) = 0$ for all $i \in V(G_1) \setminus (V_1 \cup \{2,3,4\})$. Also, we have

$$\begin{aligned}
C_i(G_1'') - C_i(G_1) &= -\frac{1}{(|V_1|+1)|V_1|/2}, \quad \forall i \in V_1\,, \\
C_2(G_1'') - C_2(G_1) &= \frac{|V_1|(|V_1|-1)/2+7}{(|V_1|+5)(|V_1|+4)/2} - \frac{(|V_1|+1)|V_1|/2+6}{(|V_1|+5)(|V_1|+4)/2} \\
&= -\frac{|V_1|-1}{(|V_1|+5)(|V_1|+4)/2}\,, \\
C_3(G_1'') - C_3(G_1) &= \frac{1}{28}\,, \\
C_4(G_1'') - C_4(G_1) &= \frac{(|V_4|+1)|V_4|/2+7}{(|V_4|+6)(|V_4|+5)/2} - \frac{(|V_4|+1)|V_4|/2+6}{(|V_4|+5)(|V_4|+4)/2} \\
&= -\frac{|V_4|(|V_4|+1)+1}{(|V_4|+6)(|V_4|+5)(|V_4|+4)/2}\,.
\end{aligned}$$

Note that $\sum_{i\in V_1}(C_i(G_1'') - C_i(G_1)) = -2/(|V_1|+1)$ and $C_2(G_1'') - C_2(G_1)$ converge to zero as $|V_1|$ goes to infinity. Similarly, $C_4(G_1'') - C_4(G_1)$ converges to zero as $|V_4|$ goes to infinity. Therefore, the quantity $\sum_{i\in V(G_1)}(C_i(G_1'') - C_i(G_1))$ is positive for sufficiently large $|V_1|$ and $|V_4|$, which means that G_1 is not a clustering coefficient locally maximizing graph in $\mathcal{G}(|V(G_1)|, |E(G_1)|)$ for sufficiently large $|V_1|$ and $|V_4|$.

As the second example, let us consider the graph $G_2 = (V(G_2), E(G_2))$ shown in Fig. 6. It is easily seen that G_2 satisfies the conditions in Theorem 2 with $K = 2$. Also, we assume that the degrees of the vertices 3, 4 and 6, which are apparently multiples of 3, are sufficiently high. Let G_2'' be the graph obtained from G_2 by removing the edge $\{1,2\}$ and adding the edge $\{4,6\}$. Then $C_i(G_2'') - C_i(G_2) = 0$ for all $i \in V(G_2) \setminus \{3,4,5,6\}$. Also, we have

$$\begin{aligned}
C_3(G_2'') - C_3(G_2) &= -\frac{1}{d_3(G_2)(d_3(G_2)-1)/2}, \\
C_4(G_2'') - C_4(G_2) &= \frac{d_4(G_2)}{(d_4(G_2)+1)d_4(G_2)/2} - \frac{d_4(G_2)}{d_4(G_2)(d_4(G_2)-1)/2} \\
&= -\frac{2}{(d_4(G_2)+1)(d_4(G_2)-1)/2}, \\
C_5(G_2'') - C_5(G_2) &= \frac{1}{15}, \\
C_6(G_2'') - C_6(G_2) &= \frac{d_6(G_2)+1}{(d_6(G_2)+1)d_6(G_2)/2} - \frac{d_6(G_2)}{d_6(G_2)(d_6(G_2)-1)/2} \\
&= -\frac{1}{d_6(G_2)(d_6(G_2)-1)/2}.
\end{aligned}$$

Note that $C_3(G_2'') - C_3(G_2)$ converges to zero as $d_3(G_2)$ goes to infinity. Similarly, $C_4(G_2'') - C_4(G_2)$ and $C_6(G_2'') - C_6(G_2)$ converge to zero as $d_4(G_2)$ and $d_6(G_2)$ go to infinity, respectively. Therefore, the quantity $\sum_{i \in V(G_1)} (C_i(G_1'') - C_i(G_1))$ is positive for sufficiently large $d_3(G_2)$, $d_4(G_2)$ and $d_6(G_2)$, which means that G_2 is not a clustering coefficient locally maximizing graph in $\mathcal{G}(|V(G_1)|, |E(G_1)|)$ for sufficiently large $d_3(G_2)$, $d_4(G_2)$ and $d_6(G_2)$.

5. Conclusions

By extending the results of Koizuka and Takahashi [25], we have given some new classes of clustering coefficient locally maximizing graphs. All graphs considered in this paper have a common property: they can be obtained from graphs by replacing each edge with a clique. The graphs obtained in this manner have a very high clustering coefficient in general. However, as we have shown in Section 4, not all of them are clustering coefficient locally maximizing graphs. This indicates that clustering coefficient locally maximizing graphs may not be characterized in a simple way. Further exploration will be needed to better understand the clustering coefficient locally maximizing graphs. Also, it is interesting to see how the results of this paper, which are based on the definition of the clustering coefficient given by Watts and Strogatz [1], change if we consider another definition [10].

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Appendix A. Proof of Theorem 2 for the Case where $K = 1$

Let $G \in \mathcal{G}(n, m)$ be any graph satisfying the conditions in Theorem 2 with $K = 1$. Let $G' \in \mathcal{G}(n, m-1)$ be the graph obtained from G by removing an edge $\{\alpha, \beta\} \in E(G)$, and let $G'' \in \mathcal{G}(n, m)$ be the graph obtained from G' by adding an edge $\{\gamma, \delta\}$ which is neither a member of $E(G')$ nor equal to $\{\alpha, \beta\}$. Because of its special structure, G has some important properties. First, for any two distinct vertices i and j, there exists at most one vertex that is adjacent to both i and j. In particular, if the vertices i and j are adjacent then there exists one and only one vertex adjacent to both i and j. Second, at least one endpoint of any edge belongs to V_0. From this property, we can hereafter assume without loss of generality that $\alpha \in V_0$. Third, $d_i(G)$ is an even number for all $i \in V(G)$.

In the following, we will show that $C(G'') - C(G) = (C(G'') - C(G')) + (C(G') - C(G)) \leq 0$ for any possible combination of α, β, γ and δ.

Let us first evaluate the quantity $C_i(G') - C_i(G)$ for each $i \in V(G)$. Let λ be the unique vertex adjacent to both α and β in G. Then $C_i(G') - C_i(G) = 0$ holds for all $i \in V(G) \setminus \{\alpha, \beta, \lambda\}$. For vertices $i \in \{\alpha, \beta\}$, we first see that $C_i(G') - C_i(G) = -1$ if $d_i(G) = 2$. We second see, by making use of Lemma 1 and (4), that $C_i(G') - C_i(G) = 0$ if $d_i(G) \geq 3$. Here it should be noted that $d_i(G) \geq 3$ implies $i \in V_0$. For the vertex λ, by making use of (5), we have

$$C_\lambda(G') - C_\lambda(G) = -\frac{1}{d_\lambda(G)(d_\lambda(G) - 1)/2} < 0\,.$$

Since we have assumed that $\alpha \in V_0$, either β or λ belongs to $V(G) \setminus V_0$, which means that either $d_\beta(G) = 2$ or $d_\lambda(G) = 2$ necessarily hold. This further implies that either $C_\beta(G') - C_\beta(G) = -1$ or $C_\lambda(G') - C_\lambda(G) = -1$ hold.

Let us next evaluate the quantity $C_i(G'') - C_i(G')$ for each $i \in V(G)$. Suppose first that G' has no vertex that is adjacent to both γ and δ. In this case, $C_i(G'') - C_i(G') = 0$ holds for all $i \in V(G) \setminus \{\gamma, \delta\}$. For vertices $i \in \{\gamma, \delta\}$, we have $C_i(G'') - C_i(G') = 0$ if $d_i(G') = 1$, and

$$\begin{aligned} C_i(G'') - C_i(G') &= \frac{t_i(G'')}{d_i(G'')(d_i(G'') - 1)/2} - \frac{t_i(G')}{d_i(G')(d_i(G') - 1)/2} \\ &= \frac{t_i(G')}{(d_i(G') + 1)d_i(G')/2} - \frac{t_i(G')}{d_i(G')(d_i(G') - 1)/2} \\ &< 0 \end{aligned}$$

if $d_i(G') \geq 2$, where we have used the equalities $d_i(G'') = d_i(G') + 1$ and $t_i(G'') = t_i(G')$. Suppose next that G' has a vertex, which is denoted by μ in the following, that is adjacent to both γ and δ. In this case, $C_i(G'') - C_i(G') = 0$ holds for all $i \in V(G) \setminus \{\gamma, \delta, \mu\}$. For vertices $i \in \{\gamma, \delta\}$, we have $C_i(G'') - C_i(G') = 1$ if $d_i(G') = 1$ and

$$\begin{aligned} C_i(G'') - C_i(G') &= \frac{t_i(G'')}{d_i(G'')(d_i(G'') - 1)/2} - \frac{t_i(G')}{d_i(G')(d_i(G') - 1)/2} \\ &= \frac{t_i(G') + 1}{(d_i(G') + 1)d_i(G')/2} - \frac{t_i(G')}{d_i(G')(d_i(G') - 1)/2} \\ &= \frac{-2t_i(G') + d_i(G') - 1}{(d_i(G') + 1)d_i(G')(d_i(G') - 1)/2} \\ &= \begin{cases} -\frac{1}{(d_i(G')+1)d_i(G')(d_i(G')-1)/2}, & \text{if } i \notin \{\alpha, \beta, \lambda\}, \\ 0, & \text{if } i \in \{\alpha, \beta\}, \\ \frac{1}{(d_i(G')+1)d_i(G')(d_i(G')-1)/2}, & \text{if } i = \lambda, \end{cases} \end{aligned} \quad \text{(A.1)}$$

if $d_i(G') \geq 2$, where the last equality follows from the relationship:

$$t_i(G') = \begin{cases} d_i(G')/2, & \text{if } i \notin \{\alpha, \beta, \lambda\}, \\ (d_i(G') - 1)/2, & \text{if } i \in \{\alpha, \beta\}, \\ d_i(G')/2 - 1, & \text{if } i = \lambda. \end{cases}$$

For the vertex μ, we have

$$C_\mu(G'') - C_\mu(G') = \frac{1}{d_\mu(G')(d_\mu(G') - 1)/2} > 0. \quad \text{(A.2)}$$

Let us finally evaluate the quantity $C(G'') - C(G) = \frac{1}{n} \sum_{i \in V(G)} (C_i(G'') - C_i(G))$. If G' has no vertex that is adjacent to both γ and δ then we can conclude that $C(G'') - C(G) \leq 0$ because we see from the analysis above that both $C_i(G') \le C_i(G)$ and $C_i(G'') \le C_i(G')$ for all $i \in V(G)$. We thus focus our attention in the following on the case where the vertex μ is adjacent to both γ and δ in G'. Here, we should note that $d_\mu(G') \ge 3$ must hold because $d_\mu(G') = 2$ implies $\{\gamma, \delta\} = \{\alpha, \beta\}$ which is a contradiction. We should also note that $d_\gamma(G') = 1$ and $d_\delta(G') = 1$ do not hold simultaneously because $d_\gamma(G') = d_\delta(G') = 1$ implies $\{\gamma, \delta\} = \{\alpha, \beta\}$ which is a contradiction. Suppose first that $d_\gamma(G') \ge 2$ and $d_\delta(G') \ge 2$. By substituting these inequalities into (A.1), we have $C_\gamma(G'') - C_\gamma(G') \le 1/3$ and $C_\delta(G'') - C_\delta(G') \le 1/3$. Also, by substituting $d_\mu(G') \ge 3$ into (A.2), we have $C_\mu(G'') - C_\mu(G') \le 1/3$. Therefore, we have

$$\begin{aligned}
&\sum_{i \in V(G)} (C_i(G'') - C_i(G)) \\
&= \sum_{i \in \{\gamma,\delta,\mu\}} (C_i(G'') - C_i(G')) + \sum_{i \in \{\alpha,\beta,\lambda\}} (C_i(G') - C_i(G)) \\
&\le \sum_{i \in \{\gamma,\delta,\mu\}} (C_i(G'') - C_i(G')) - 1 \\
&\le \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - 1 \\
&= 0\,.
\end{aligned}$$

Suppose next that either $d_\gamma(G') = 1$ or $d_\delta(G') = 1$ holds. In the following, we assume without loss of generality that $d_\gamma(G') = 1$ and $d_\delta(G') \ge 2$. Then it is easily seen that $d_\gamma(G) = 2$, $\gamma \in \{\alpha, \beta\}$, $\mu = \lambda$ and $\delta \notin \{\alpha, \beta\}$. Therefore, we have

$$\begin{aligned}
&\sum_{i \in V(G)} (C_i(G'') - C_i(G)) \\
&= \sum_{i \in \{\gamma,\delta,\mu\}} (C_i(G'') - C_i(G')) + \sum_{i \in \{\alpha,\beta,\lambda\}} (C_i(G') - C_i(G)) \\
&\le \sum_{i \in \{\gamma,\delta,\mu\}} (C_i(G'') - C_i(G')) + \sum_{i \in \{\gamma,\mu\}} (C_i(G') - C_i(G)) \\
&= 1 - \frac{1}{(d_\delta(G')+1)d_\delta(G')(d_\delta(G')-1)/2} + \frac{1}{d_\mu(G')(d_\mu(G')-1)/2} \\
&\quad - 1 - \frac{1}{d_\mu(G)(d_\mu(G)-1)/2} \\
&= -\frac{1}{(d_\delta(G')+1)d_\delta(G')(d_\delta(G')-1)/2}
\end{aligned}$$

$$< \quad 0$$

where we have used the equality $d_\mu(G') = d_\mu(G)$. $\square$